

# POINTWISE ESTIMATES FOR SPLINE GRAM MATRIX INVERSES OF ORDER THREE

MARKUS PASSENBRUNNER

**ABSTRACT.** In this article we prove a pointwise estimate for inverses of Gram matrices corresponding to spline systems of order three. We use this inequality to obtain that the orthogonal projection  $P_\Delta f$  of  $f \in L^1[0, 1]$  onto the spline space of order 3 with mesh points  $\Delta$  converges to  $f$  almost everywhere, provided the maximal mesh width of  $\Delta$  tends to zero.

## CONTENTS

1. Introduction	1
2. Preliminaries	2
2.1. The Sherman-Morrison formula	2
2.2. B-splines	3
2.3. Total positivity	5
3. The main result	5
4. An iteration formula for inverses of matrices	6
4.1. Tridiagonal matrices	8
4.2. 2-band matrices	8
5. Piecewise linear splines	9
6. Piecewise quadratic splines	12
6.1. The Gram matrix	12
6.2. Continuous piecewise quadratic splines	14
6.3. Continuously differentiable piecewise quadratic splines	18
7. Applications	24
References	31

## 1. INTRODUCTION

Let  $k \in \mathbb{N}$  and  $A = (\langle N_{i,k}, N_{j,k} \rangle)_{i,j=1}^M$  be the Gram matrix of the B-spline functions  $N_{i,k}$ ,  $1 \leq i \leq M$ , of order  $k$  corresponding to an arbitrary partition  $\Delta$  of the unit interval  $[0, 1]$ . For the exact definition of B-splines, see Section 2. Moreover, let  $P_\Delta^k$  be the orthogonal projection operator from  $L^2[0, 1]$  onto the space spanned by the spline functions  $(N_{i,k})_{i=1}^M$ .

Exponential decay inequalities (away from the main diagonal) for the matrix  $(b_{i,j})_{i,j} := A^{-1}$  translate into boundedness properties of  $P_\Delta^k$ . For instance, the

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inequality

$$(1.1) \quad |b_{i,j}| \leq \frac{Cq^{|i-j|}}{\max\{|\operatorname{supp} N_{i,k}|, |\operatorname{supp} N_{j,k}|\}}$$

for some constants  $C, 0 < q < 1$  (uniform in  $\Delta$ ), is equivalent to the uniform boundedness of  $P_\Delta^k$  as an operator from  $L^\infty[0, 1]$  to  $L^\infty[0, 1]$ . (cf. [Cie00]) In fact, Shadrin showed in his paper [Sha01] that for each fixed  $k$ , this uniform boundedness of  $P_\Delta^k$  holds, implying (1.1). This additionally yields that for every partition  $\Delta$ , the corresponding orthogonal spline system is a basis in  $C[0, 1]$  (and  $L^p[0, 1]$  for  $1 \leq p < \infty$ ).

In order to show more properties concerning convergence of orthogonal spline series (like almost everywhere convergence or unconditional convergence), it is natural to improve (1.1) in a way that the desired convergence property is implied. For example, a uniform inequality of the form

$$(1.2) \quad |b_{i,j}| \leq \frac{Cq^{|i-j|}}{\max_{i \wedge j \leq l \leq i \vee j} |\operatorname{supp} N_{l,k}|}$$

is used in [CK97] and [GK04] to prove convergence almost everywhere (for weak Lebesgue points) and unconditional convergence in  $L^p$  for  $1 < p < \infty$  of general Franklin series (that is orthogonal piecewise linear splines).

The aim of this paper is to prove the uniform inequality (1.2) for piecewise quadratic splines ( $k = 3$ ). As a first application of this result, we show the almost everywhere convergence of  $P_\Delta f$  to  $f$  for  $f \in L^1$ , provided the maximal mesh width of  $\Delta$  tends to zero. The main ingredients of the proof of (1.2) are

- (1) The Sherman-Morrison formula applied to rank-2-updates of matrices and
- (2) the total positivity of the matrix  $A$ .

The organization of this article is as follows: In Section 2 we recall some preliminaries that include the Sherman-Morrison formula for inverse matrices, the definition of B-spline functions and some results concerning total positivity of B-spline Gram matrices. Section 3 contains the formulation of our main theorem, that is the exact form in which (1.2) holds. In Section 4, we use the Sherman-Morrison formula to derive an expression for iterated calculation of inverse Gram matrices of B-splines, which is the main starting point of the proof of (1.2). In Section 5, we apply our method of proof to obtain the known inequality (1.2) for piecewise linear splines. The purpose of this presentation is to illustrate the basic steps that are essentially the same as in more general cases. In Section 6, we derive our main result (1.2) for piecewise quadratic splines. Section 7 gives a few applications of this result, including almost everywhere convergence of  $P_\Delta f$  to  $f$  for  $f \in L^1[0, 1]$  as the maximal mesh width converges to zero.

## 2. PRELIMINARIES

**2.1. The Sherman-Morrison formula.** We have the following formula for the inverse of a rank  $m$  perturbation of a given matrix  $A$ .

**Theorem 2.1.** [SM50, Woo50] *Let  $A$  be an invertible  $n \times n$  matrix and  $U, V$   $n \times m$  matrices. If the  $m \times m$  matrix*

$$1 + V^T A^{-1} U$$

is invertible, then we have

$$(A + UV^T)^{-1} = A^{-1} - A^{-1}U(1 + V^T A^{-1}U)^{-1}V^T A^{-1}.$$

**Remark 2.2.** In applications of this formula,  $m$  is typically much smaller than  $n$ . We will apply it for the choice  $m = 2$ .

**2.2. B-splines.** Suppose we are given a partition

$$(2.1) \quad \Delta = (s_i)_{i=1}^L, \quad 0 = s_1 < s_2 < \cdots < s_{L-1} < s_L = 1$$

of the unit interval  $[0, 1]$  and an integer  $k \geq 1$ . We denote by  $P_k(\Delta)$  the space of piecewise polynomials of order  $k$  with respect to the partition  $\Delta$ , i.e.

$$(2.2) \quad P_k(\Delta) := \{f \in \mathbb{R}^{[0,1]} : f|_{(s_i, s_{i+1})} \text{ is a polynomial of degree } k-1 \\ \text{for each } 1 \leq i \leq L-1\}.$$

Furthermore, we define the spline space

$$(2.3) \quad S_k(\Delta, \sigma) := \{f \in P_k(\Delta) : f \in C^{\sigma_j}(s_j) \text{ for all } 2 \leq j \leq L-1\},$$

where  $(\sigma_j)_{j=2}^{L-1}$  is a sequence of integers (smoothness parameters) such that  $0 \leq \sigma_j \leq k-2$  for all  $j$ . Corresponding to the partition  $\Delta$ , we define an extended sequence of points  $\bar{\Delta}$  where we allow each knot  $s_i$  to occur more than once. For this, let  $\mu = (\mu_1, \dots, \mu_L)$  with  $\mu_1 = \mu_L = k$  and  $1 \leq \mu_i \leq k-1$  for  $2 \leq i \leq L-1$  be a multiplicity vector. We denote the  $\ell^1$ -norm of a vector by  $|\cdot|$ . Then we define  $\bar{\Delta} = \bar{\Delta}(\Delta, \mu)$  to be the unique sequence of points  $(t_i)_{i=1}^{|\mu|}$  that satisfies

$$|\{1 \leq i \leq |\mu| : t_i = s_j\}| = \mu_j \quad \text{for all } 1 \leq j \leq L \quad \text{and} \\ t_i \leq t_{i+1} \quad \text{for all } 1 \leq i \leq |\mu| - 1.$$

So,  $\bar{\Delta}$  is the ordered sequence of points in  $\Delta$  where each point  $s_j$  occurs  $\mu_j$  times. Corresponding to the sequence  $\bar{\Delta} = (t_i)_{i=1}^{|\mu|}$ , we define the B-splines  $N_{i,k}$  of order  $k$  as

$$(2.4) \quad N_{i,k}(x) = (-1)^k (t_{i+k} - t_i) [t_i, \dots, t_{i+k}](x - \cdot)_+^{k-1}, \quad 1 \leq i \leq |\mu| - k + 1 =: M,$$

where  $[t_i, \dots, t_{i+k}]$  is the standard (forward) divided difference operator with respect to the points  $t_i, \dots, t_{i+k}$  and  $u_+^k$  is the truncated power function defined as

$$u_+^0 := \chi_{[0, \infty)}(u), \quad u_+^k := u^k u_+^0 \text{ for } k \geq 1.$$

We remark that  $N_{i,1}$  is the characteristic function  $\chi_{[t_i, t_{i+1})}$  of the interval  $[t_i, t_{i+1})$ . The relation between  $S_k(\Delta, \sigma)$  and the defined B-splines corresponding to the sequence of points  $\bar{\Delta}(\Delta, \mu)$  is given by the following well known theorem, see for instance [Sch81], Theorem 4.9.

**Theorem 2.3.** *Let  $k \in \mathbb{N}$ ,  $\Delta$  as in (2.1) and  $\sigma$  a sequence of smoothness parameters as above. Define the sequence  $\mu = (k, k-1-\sigma_2, \dots, k-1-\sigma_{L-1}, k)$  and let  $N_{i,k}$  be the B-splines corresponding to the extended sequence of points  $\bar{\Delta}(\Delta, \mu)$  defined above. Then*

$$\text{span}\{N_{i,k}\}_{i=1}^M = S_k(\Delta, \sigma),$$

*i.e. the B-splines are a basis in the space of piecewise polynomials with certain prescribed smoothness at the mesh points.*

We collect classical properties of the B-splines  $N_{i,k}$  in the following

**Proposition 2.4.** *The B-splines  $N_{i,k}$ ,  $k \geq 1$ , admit the following properties.*

- (1)  $N_{i,k}(x) \geq 0$  for all  $x \in \mathbb{R}$ ,
- (2)  $\text{supp } N_{i,k} = [t_i, t_{i+k}]$ ,
- (3)  $\sum_{i=1}^M N_{i,k}(x) = 1$  for all  $x \in [0, 1]$ ,
- (4) the recursion formula for B-splines:

$$N_{i,k+1} = \frac{\cdot - t_i}{t_{i+k} - t_i} N_{i,k} + \frac{t_{i+k+1} - \cdot}{t_{i+k+1} - t_{i+1}} N_{i+1,k},$$

- (5) the  $L^1$ -norm of  $N_{i,k}$  is given by

$$\|N_{i,k}\|_1 = \int_0^1 N_{i,k}(x) dx = \frac{t_{i+k} - t_i}{k}.$$

In order to keep future expressions involving distances of points in  $\overline{\Delta}$  simple, we introduce the following notation.

**Definition 2.5.** Let  $k \in \mathbb{N}$ ,  $\Delta$  given as in (2.1),  $\mu$  a multiplicity vector and  $\overline{\Delta} = \overline{\Delta}(\Delta, \mu) = \{t_i\}_{i=1}^{|\mu|}$  be the extended sequence of points as above. Furthermore, let  $t_i = 0$  for  $i \leq 0$  and  $t_i = 1$  for  $i \geq |\mu| + 1$ . Then we define

$$(2.5) \quad (m, n)_j := (mn)_j := t_{j+m} - t_{j+n}$$

for integer parameters  $m, n, j$ . We define  $\eta : \{1, \dots, M\} \times \{1, \dots, M\} \rightarrow \mathbb{R}_+$  to be an arbitrary function that satisfies the following axioms:

- (1)  $\eta(i, i) = (k, 0)_i$  for all  $1 \leq i \leq M$ ,
- (2)  $\eta(i, j) = \eta(j, i)$  for all  $1 \leq i, j \leq M$ ,
- (3)  $\eta(j, n) \leq \eta(j, n+1) \leq \eta(j, n) \frac{(k, 0)_{n+1}}{(k-1, 0)_{n+1}}$  for all  $1 \leq n \leq M-1, 1 \leq j \leq n$ .

**Example 2.6.** (1) The function

$$(2.6) \quad \eta_1(i, j) := \max_{i \wedge j \leq l \leq i \vee j} (k, 0)_l$$

satisfies the axioms (1)–(3) of the above definition. Indeed, axioms (1) and (2) and the left hand inequality of axiom (3) are trivially true for  $\eta_1$ . The right hand inequality of axiom (3) is an immediate consequence of the two obvious inequalities

$$(k-1, 0)_{n+1} \leq (k, 0)_{n+1}, \quad (k-1, 0)_{n+1} \leq (k, 0)_n \leq \eta_1(j, n)$$

and the definition of  $\eta_1$ .

- (2) The function

$$(2.7) \quad \eta_2(i, j) := t_{i \vee j+k} - t_{i \wedge j}$$

satisfies axioms (1)–(3) of Definition 2.5. Axioms (1), (2) and the left hand inequality of axiom (3) are again immediate consequences of the definition. The right hand inequality of property (3) follows from the elementary inequality

$$\frac{a}{(a+b)(a+c)} \leq \frac{1}{a+b+c}$$

for positive real numbers  $a, b, c$ .

Clearly,  $\eta_1(i, j) \leq \eta_2(i, j) \leq |i-j|\eta_1(i, j)$  for all  $1 \leq i, j \leq M$ .

**Remark 2.7.** Since the values for  $m, n$  in the expressions  $(m, n)_j$  are not greater than 6 in our cases, we typically write for instance  $(10)_i$  to denote  $(1, 0)_i$ .

**2.3. Total positivity.** We denote by  $Q_{m,n}$  the set of strictly increasing sequences of  $m$  integers from the set  $\{1, \dots, n\}$ . Let  $A$  be an  $n \times n$ -matrix. For  $\alpha, \beta \in Q_{m,n}$ , we denote by  $A[\alpha; \beta]$  the submatrix of  $A$  consisting of the rows indexed by  $\alpha$  and the columns indexed by  $\beta$ . Furthermore we let  $\alpha'$  (the complement of  $\alpha$ ) be the uniquely determined element of  $Q_{n-m,n}$  that consists of all integers in  $\{1, \dots, n\}$  not occurring in  $\alpha$ . In addition, we use the notation  $A(\alpha; \beta) := A[\alpha'; \beta']$ . This notation is taken from [And87].

**Definition 2.8.** Let  $A$  be an  $n \times n$ -matrix.  $A$  is called *totally positive*, if

$$(2.8) \quad \det A[\alpha; \beta] \geq 0, \quad \text{for } \alpha, \beta \in Q_{m,n}, 1 \leq m \leq n.$$

The cofactor formula  $b_{i,j} = (-1)^{i+j} \det A(j; i) / \det A$  for the inverse  $B = (b_{i,j})_{i,j=1}^M$  of the matrix  $A$  leads to

**Proposition 2.9.** *Inverses  $B = (b_{i,j})$  of totally positive matrices  $A = (a_{i,j})$  have the checkerboard property. This means that*

$$(-1)^{i+j} b_{i,j} \geq 0 \quad \text{for all } i, j.$$

We introduce the notation  $\langle f, g \rangle := \int_0^1 f \bar{g}$  for the standard inner product of two  $L^2$  functions on the unit interval  $[0, 1]$  with respect to Lebesgue measure.

**Remark 2.10.** The Gram matrix of B-splines  $A = (\langle N_{i,k}, N_{j,k} \rangle)_{i,j=1}^M$  of arbitrary order  $k$  and arbitrary partition  $\bar{\Delta}$  is totally positive (see [dB68]). This is a consequence of the so called basic composition formula (Equation (2.5), Chapter 1 in [Kar68]) and the fact that the kernel  $N_{i,k}(x)$ , depending on the variables  $i$  and  $x$ , is totally positive (Theorem 4.1, Chapter 10 in [Kar68]). Thus the inverse of  $A$  possesses the checkerboard property by the above proposition.

### 3. THE MAIN RESULT

Our main result is about the inverse Gramian of piecewise quadratic splines.

**Main Theorem 3.1.** *Let  $k = 3$ ,  $\Delta$  be an arbitrary partition of the unit interval  $[0, 1]$  and  $\mu$  an arbitrary multiplicity vector. Moreover, let  $\eta$  be any function as in Definition 2.5. Then, the inverse  $B = (b_{i,j})_{i,j=1}^M$  of the Gram matrix  $A = (\langle N_{i,3}, N_{j,3} \rangle)_{i,j=1}^M$  satisfies the bound*

$$(3.1) \quad |b_{i,j}| \leq C_1 \frac{q^{|i-j|}}{\eta(i,j)} \quad \text{for } 1 \leq i, j \leq M,$$

where  $0 < q < 1$  and  $C_1 > 0$  are constants independent of  $i, j$  and  $\Delta$ . They can be chosen as  $q = (87/100)^{1/2}$  and  $C_1 = C(1 + \frac{12}{75} \frac{C}{1-q^2})$ , where  $C = 12q^{-2}(\frac{6}{5})^2$ .

**Remark 3.2.** It was proved by Shadrin [Sha01] that the  $L^\infty$ -norm of the  $L^2$ -spline projector is bounded independently of the knot sequence for each  $k \in \mathbb{N}$ . For this assertion there are a number of equivalent properties in [Cie00]. One of

them is that the inverse  $B = (b_{i,j})_{i,j=1}^M$  of the Gram matrix  $A = (\langle N_{i,k}, N_{j,k} \rangle)_{i,j=1}^M$  satisfies an inequality of the form

$$(3.2) \quad |b_{i,j}| \leq \frac{c_k q_k^{|i-j|}}{|\text{supp } N_{i,k}| \vee |\text{supp } N_{j,k}|} \quad \text{for } 1 \leq i, j \leq M,$$

where the constants  $c_k, 0 < q_k < 1$  depend only on the spline order  $k$ . The result of Theorem 3.1 is thus an improvement of this inequality for  $k = 3$ , since in  $\eta_1(i, j) = \max_{i \wedge j \leq l \leq i \vee j} |\text{supp } N_{l,k}|$ , the maximum ranges over all indices *between*  $i$  and  $j$  in contrast to the maximum in (3.2). For an analogous result in the case  $k = 2$ , cf. Remark 5.4, [Cie66] and [KS89].

The first tool for the proof of Theorem 3.1 is an iteration formula for inverse matrices presented in Section 4. To illustrate the method of proof, we first show an analogous estimate for piecewise linear splines in Section 5. The next step is to consider piecewise quadratic splines, that solely have double points, which is done in Section 6.2. This limiting case also provides some insight in the general case of piecewise quadratic splines. Finally, Theorem 3.1 is a corollary to Theorem 6.17 proved in Section 6.3.

#### 4. AN ITERATION FORMULA FOR INVERSES OF MATRICES

Let  $A = (a_{i,j})_{i,j=1}^M$  for some  $M \in \mathbb{N}$  be an invertible  $M \times M$ -matrix and define

$$A_n = (a_{i,j})_{i,j=1}^n \quad \text{for } 1 \leq n \leq M$$

to be the  $n \times n$  matrix consisting of the topmost  $n$  rows and the leftmost  $n$  columns of  $A$ . Furthermore set  $B_n := A_n^{-1}$  if  $A_n$  is invertible and let  $B_n = (b_{i,j}^n)_{i,j=1}^n$ . Then  $A_{n+1}$  may be written as the sum of two matrices as follows.

$$(4.1) \quad A_{n+1} = \begin{pmatrix} A_n & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & a_{n+1,n+1} \end{pmatrix} + \begin{pmatrix} \mathbf{0}_{n \times n} & u^n \\ (v^n)^T & 0 \end{pmatrix} =: S_n + T_n,$$

where  $v^n \in \mathbb{R}^n$  is the column vector  $(a_{n+1,1}, \dots, a_{n+1,n})^T$  and  $u^n$  is the column vector  $(a_{1,n+1}, \dots, a_{n,n+1})^T$ .  $T_n$  is a rank 2 matrix that can be written as the product  $T_n = U_n V_n^T$  where  $U_n$  and  $V_n$  are  $(n+1) \times 2$  matrices and defined as

$$U_n = \begin{pmatrix} u^n & \mathbf{0}_{n \times 1} \\ 0 & 1 \end{pmatrix}, \quad V_n = \begin{pmatrix} \mathbf{0}_{n \times 1} & v^n \\ 1 & 0 \end{pmatrix}.$$

Theorem 2.1 and the above decomposition (4.1) of  $A_{n+1}$  in two summands yield the following

**Corollary 4.1.** *Let  $1 \leq n \leq M$ . Additionally, suppose that  $A_n, B_n$  and  $v^n, u^n$  are defined as above and set  $v = v^n, u = u^n$ . If  $A_n$  is invertible,  $a_{n+1,n+1} \neq 0$  and  $a_{n+1,n+1} - v^T B_n u \neq 0$  we have the following formula for  $B_{n+1} = A_{n+1}^{-1}$ :*

$$B_{n+1} = \begin{pmatrix} B_n & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & 0 \end{pmatrix} + \frac{1}{a_{n+1,n+1} - v^T B_n u} \begin{pmatrix} B_n u v^T B_n & -B_n u \\ -v^T B_n & 1 \end{pmatrix},$$

*Proof.* We compute the matrix products occurring in the Sherman-Morrison formula step by step and conclude

$$S_n^{-1} U_n = \begin{pmatrix} B_n & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & a_{n+1,n+1}^{-1} \end{pmatrix} \begin{pmatrix} u & \mathbf{0}_{n \times 1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} B_n u & \mathbf{0}_{n \times 1} \\ 0 & a_{n+1,n+1}^{-1} \end{pmatrix},$$

$$\begin{aligned}
V_n^T S_n^{-1} U_n &= \begin{pmatrix} \mathbf{0}_{1 \times n} & 1 \\ v^T & 0 \end{pmatrix} \begin{pmatrix} B_n u & \mathbf{0}_{n \times 1} \\ 0 & a_{n+1,n+1}^{-1} \end{pmatrix} = \begin{pmatrix} 0 & a_{n+1,n+1}^{-1} \\ v^T B_n u & 0 \end{pmatrix}, \\
(1 + V_n^T S_n^{-1} U_n)^{-1} &= \begin{pmatrix} 1 & a_{n+1,n+1}^{-1} \\ v^T B_n u & 1 \end{pmatrix}^{-1} \\
&= \frac{1}{1 - a_{n+1,n+1}^{-1} v^T B_n u} \begin{pmatrix} 1 & -a_{n+1,n+1}^{-1} \\ -v^T B_n u & 1 \end{pmatrix}.
\end{aligned}$$

Define  $x := (1 - a_{n+1,n+1}^{-1} v^T B_n u)^{-1}$ . We get further

$$\begin{aligned}
U_n(1 + V_n^T S_n^{-1} U_n)^{-1} &= x \begin{pmatrix} u & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & 1 \end{pmatrix} \begin{pmatrix} 1 & -a_{n+1,n+1}^{-1} \\ -v^T B_n u & 1 \end{pmatrix} \\
&= x \begin{pmatrix} u & -a_{n+1,n+1}^{-1} u \\ -v^T B_n u & 1 \end{pmatrix}, \\
U_n(1 + V_n^T S_n^{-1} U_n)^{-1} V_n^T &= x \begin{pmatrix} u & -a_{n+1,n+1}^{-1} u \\ -v^T B_n u & 1 \end{pmatrix} \begin{pmatrix} \mathbf{0}_{1 \times n} & 1 \\ v^T & 0 \end{pmatrix} \\
&= x \begin{pmatrix} -a_{n+1,n+1}^{-1} u v^T & u \\ v^T & -v^T B_n u \end{pmatrix}, \\
U_n(1 + V_n^T S_n^{-1} U_n)^{-1} V_n^T S_n^{-1} &= x \begin{pmatrix} -a_{n+1,n+1}^{-1} u v^T & u \\ v^T & -v^T B_n u \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} B_n & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & a_{n+1,n+1}^{-1} \end{pmatrix} \\
&= x \begin{pmatrix} -a_{n+1,n+1}^{-1} u v^T B_n & a_{n+1,n+1}^{-1} u \\ v^T B_n & -a_{n+1,n+1}^{-1} v^T B_n u \end{pmatrix}, \\
S_n^{-1} U_n(1 + V_n^T S_n^{-1} U_n)^{-1} V_n^T S_n^{-1} &= x \begin{pmatrix} B_n & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & a_{n+1,n+1}^{-1} \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} -a_{n+1,n+1}^{-1} u v^T B_n & a_{n+1,n+1}^{-1} u \\ v^T B_n & -a_{n+1,n+1}^{-1} v^T B_n u \end{pmatrix} \\
&= a_{n+1,n+1}^{-1} x \begin{pmatrix} -B_n u v^T B_n & B_n u \\ v^T B_n & -a_{n+1,n+1}^{-1} v^T B_n u \end{pmatrix}.
\end{aligned}$$

If we insert these expressions in the Sherman-Morrison-formula, Theorem 2.1, we conclude

$$\begin{aligned}
B_{n+1} &= \begin{pmatrix} B_n & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & a_{n+1,n+1}^{-1} \end{pmatrix} + \frac{1}{a_{n+1,n+1} - v^T B_n u} \begin{pmatrix} B_n u v^T B_n & -B_n u \\ -v^T B_n & a_{n+1,n+1}^{-1} v^T B_n u \end{pmatrix} \\
&= \begin{pmatrix} B_n & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & 0 \end{pmatrix} + \frac{1}{a_{n+1,n+1} - v^T B_n u} \begin{pmatrix} B_n u v^T B_n & -B_n u \\ -v^T B_n & 1 \end{pmatrix}.
\end{aligned}$$

This proves the corollary.  $\square$

We now employ this corollary in the cases where the matrix  $A$  is a symmetric tridiagonal or a symmetric 2-band matrix.

**4.1. Tridiagonal matrices.** Let  $A$  be a symmetric tridiagonal  $M \times M$ -matrix. Using Corollary 4.1 on  $A$ , we get for  $1 \leq n \leq M - 1$

$$(4.2) \quad B_{n+1} = \begin{pmatrix} B_n & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & 0 \end{pmatrix} + (a_{n+1,n+1} - b_{n,n}^n a_{n,n+1}^2)^{-1} C_{n+1},$$

where  $C_{n+1}$  is given by

$$(C_{n+1})_{i,j} = \begin{cases} b_{i,n}^n b_{j,n}^n a_{n,n+1}^2 & \text{if } 1 \leq i, j \leq n, \\ -b_{i,n}^n a_{n,n+1} & \text{if } 1 \leq i \leq n, j = n+1, \\ -b_{j,n}^n a_{n,n+1} & \text{if } 1 \leq j \leq n, i = n+1, \\ 1 & \text{if } i = j = n+1. \end{cases}$$

**4.2. 2-band matrices.** Let  $A = (a_{i,j})$  be a symmetric 2-band matrix. This means that  $a_{i,j} = 0$  for  $|i - j| > 2$ . Using Corollary 4.1 on  $A$ , we get for  $2 \leq n \leq M - 1$

$$(4.3) \quad B_{n+1} = \begin{pmatrix} B_n & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & 0 \end{pmatrix} + b_{n+1,n+1}^{n+1} C_{n+1},$$

where

$$(4.4) \quad b_{n+1,n+1}^{n+1} = (a_{n+1,n+1} - b_{n,n}^n a_{n,n+1}^2 - 2b_{n-1,n}^n a_{n-1,n+1} a_{n,n+1} - b_{n-1,n-1}^n a_{n-1,n+1}^2)^{-1}$$

and  $C_{n+1}$  is given by

$$(4.5) \quad (C_{n+1})_{i,j} = \begin{cases} -b_{i,n}^n a_{n,n+1} - b_{i,n-1}^n a_{n-1,n+1} & \text{if } 1 \leq i \leq n, j = n+1, \\ -b_{j,n}^n a_{n,n+1} - b_{j,n-1}^n a_{n-1,n+1} & \text{if } 1 \leq j \leq n, i = n+1, \\ (C_{n+1})_{i,n+1} (C_{n+1})_{j,n+1} & \text{if } 1 \leq i, j \leq n, \\ 1 & \text{if } i = j = n+1. \end{cases}$$

The following two lemmas are independent of the special form of matrices considered in the next sections. We will use them in Section 6.3 to estimate inverses of Gram matrices corresponding to splines of order 3.

**Lemma 4.2.** *Let  $A$  be a symmetric 2-banded  $M \times M$ -matrix such that  $B_n = A_n^{-1}$  is checkerboard for  $1 \leq n \leq M$ . Then, the inequality*

$$(4.6) \quad b_{n+1,n+1}^{n+1} \leq \left( a_{n+1,n+1} - b_{n,n}^n a_{n,n+1} \left( a_{n,n+1} - \frac{2a_{n,n} a_{n-1,n+1}}{a_{n-1,n}} \right) - 2 \frac{a_{n,n+1} a_{n-1,n+1}}{a_{n-1,n}} - a_{n-1,n+1}^2 b_{n-1,n-1}^{n-1} (1 + b_{n,n}^n b_{n-1,n-1}^{n-1} a_{n-1,n}^2) \right)^{-1}$$

holds for  $2 \leq n \leq M - 1$ .

*Proof.* By (4.4) and the checkerboard property of  $B_n$ ,  $b_{n+1,n+1}^{n+1}$  is given by

$$(4.7) \quad b_{n+1,n+1}^{n+1} = (a_{n+1,n+1} - b_{n,n}^n a_{n,n+1}^2 + 2|b_{n-1,n}^n| a_{n,n+1} a_{n-1,n+1} - b_{n-1,n-1}^n a_{n-1,n+1}^2)^{-1}.$$

Another consequence of the identities (4.3)–(4.5) is

$$b_{i,j}^n = b_{i,j}^{n-1} + b_{n,n}^n (b_{i,n-2}^{n-1} a_{n-2,n} + b_{i,n-1}^{n-1} a_{n-1,n}) (b_{j,n-2}^{n-1} a_{n-2,n} + b_{j,n-1}^{n-1} a_{n-1,n})$$



for  $1 \leq i, j \leq n-1$ . The choice  $i = j = n-1$  in these equations together with the checkerboard property of  $B_n$  yield the estimate

$$(4.8) \quad b_{n-1,n-1}^n \leq b_{n-1,n-1}^{n-1} (1 + b_{n,n}^n b_{n-1,n-1}^{n-1} a_{n-1,n}^2).$$

The defining property of the inverse matrix  $B_n = A_n^{-1}$ , the fact that  $A_n$  is 2-banded and the checkerboard property of  $B_n$  again imply

$$(4.9) \quad \begin{aligned} |b_{n-1,n}^n| &= a_{n-1,n}^{-1} (b_{n,n}^n a_{n,n} + b_{n-2,n}^n a_{n-2,n} - 1) \\ &\geq a_{n-1,n}^{-1} (b_{n,n}^n a_{n,n} - 1). \end{aligned}$$

Finally, inserting (4.8) and (4.9) in (4.7) yields the conclusion of the lemma.  $\square$

**Remark 4.3.** The crucial fact about the foregoing lemma is that inequality (4.6) depends only on the matrix  $A$  and on the rightmost and bottommost entries  $b_{n,n}^n, b_{n-1,n-1}^{n-1}$  of the matrices  $B_n, B_{n-1}$  respectively.

**Lemma 4.4.** *Let  $A$  be as in Lemma 4.2 and  $2 \leq n \leq M$ . Then, if  $1 \leq j \leq n-1$ , the estimate*

$$(4.10) \quad |b_{j,n}^n| \leq |b_{j,n-1}^{n-1}| b_{n,n}^n a_{n-1,n}$$

*holds. Additionally for  $3 \leq n \leq M$  and  $1 \leq j \leq n-2$  we have*

$$(4.11) \quad |b_{j,n}^n| \leq |b_{j,n-1}^{n-1}| b_{n,n}^n \left( a_{n-1,n} - \frac{a_{n-2,n} a_{n-1,n-1}}{a_{n-2,n-1}} \right).$$

*Proof.* First, we note that (4.3)–(4.5) yield

$$(4.12) \quad b_{j,n}^n = -b_{n,n}^n (b_{j,n-2}^{n-1} a_{n-2,n} + b_{j,n-1}^{n-1} a_{n-1,n}) \quad \text{if } 1 \leq j \leq n-1.$$

Since  $B_n$  is checkerboard, (4.10) follows for  $1 \leq j \leq n-1$ . By the defining property of  $B_n = A_n^{-1}$  we obtain

$$(4.13) \quad b_{j,n-3}^{n-1} a_{n-3,n-1} + b_{j,n-2}^{n-1} a_{n-2,n-1} + b_{j,n-1}^{n-1} a_{n-1,n-1} = \delta_{j,n-1}.$$

Thus we get, under the assumption  $1 \leq j \leq n-2$ ,

$$(4.14) \quad \begin{aligned} |b_{j,n-2}^{n-1}| &= a_{n-2,n-1}^{-1} (a_{n-3,n-1} |b_{j,n-3}^{n-1}| + a_{n-1,n-1} |b_{j,n-1}^{n-1}|) \\ &\geq a_{n-2,n-1}^{-1} a_{n-1,n-1} |b_{j,n-1}^{n-1}|, \end{aligned}$$

since  $B_{n-1}$  is checkerboard. Now use the checkerboard property of  $B_n$  in (4.12) and insert estimate (4.14) in (4.12) to conclude the assertion of the lemma.  $\square$

## 5. PIECEWISE LINEAR SPLINES

We now apply Corollary 4.1 to the case of piecewise linear continuous splines to get geometric estimates for the entries of their Gram matrix inverse. We let  $\Delta = (s_i)_{i=1}^L$  be a partition of the unit interval  $[0, 1]$  as in (2.1) and set the  $k = 2$ . The multiplicity vector  $\mu$  in the case for piecewise linear continuous splines is given by

$$\mu = (2, 1, 1, \dots, 1, 1, 2).$$

We construct the extended sequence of points  $\bar{\Delta} = \bar{\Delta}(\Delta, \mu) = (t_i)_{i=1}^{|\mu|}$  as in Section 2.2 and note that in the underlying case,  $M := |\mu| - k = L$ . Furthermore, we construct the B-spline functions  $N_i \equiv N_{i,2}$  for  $1 \leq i \leq M$  as in Section 2.2;

that is in this section  $N_i$  is the unique piecewise linear continuous function on the unit interval  $[0, 1]$  such that

$$N_i(s_j) = \delta_{i,j} \quad \text{for } 1 \leq j \leq L.$$

Now we consider the Gram matrix  $A = (a_{i,j})_{i,j=1}^M = (\langle N_i, N_j \rangle)_{i,j=1}^M$  obtained from these functions. Using the special form of the piecewise linear B-splines  $N_i$ , we get

$$(5.1) \quad a_{i,i} = (20)_i/3 \quad \text{if } 1 \leq i \leq M,$$

$$(5.2) \quad a_{i,i+1} = a_{i+1,i} = (21)_i/6 \quad \text{if } 1 \leq i \leq M-1,$$

$$(5.3) \quad a_{i,j} = 0 \quad \text{if } |i-j| > 1.$$

Note that  $A$  is symmetric and tridiagonal, therefore we may apply the formulas of Section 4.1 to  $A$ . If we do and insert the expressions (5.1)–(5.3) for the Gram matrix  $A$ , we obtain

$$(5.4) \quad B_{n+1} = \begin{pmatrix} B_n & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & 0 \end{pmatrix} + 3 \left( (20)_{n+1} - \frac{1}{12} b_{n,n}^n (21)_n^2 \right)^{-1} C_{n+1},$$

where

$$(5.5) \quad (C_{n+1})_{i,j} = \begin{cases} b_{i,n}^n b_{j,n}^n (21)_n^2 / 36 & \text{if } 1 \leq i, j \leq n, \\ -b_{i,n}^n (21)_n / 6 & \text{if } 1 \leq i \leq n, j = n+1, \\ -b_{j,n}^n (21)_n / 6 & \text{if } 1 \leq j \leq n, i = n+1, \\ 1 & \text{if } i = j = n+1. \end{cases}$$

**Lemma 5.1.** *Let the matrix  $A$  be as above and  $1 \leq n \leq M$ . Then we have the estimates*

$$(5.6) \quad \frac{3}{(20)_n} \leq b_{n,n}^n \leq \frac{3}{\frac{3}{4}(10)_n + (21)_n} \leq \frac{4}{(20)_n}$$

for the rightmost bottommost element  $b_{nn}^n$  of the matrix  $B_n = A_n^{-1}$ .

*Proof.* We proceed by induction on  $n$ . For  $n = 1$ , we have by definition of  $B_1$  and (5.1)

$$\frac{3}{(20)_1} = b_{1,1}^1 = 3 \left( \frac{3}{4}(10)_1 + (21)_1 \right)^{-1},$$

since  $(10)_1 = 0$  and thus equality on both sides of (5.6). We know from Remark 2.10 that  $A$  is totally positive, so a fortiori  $A_n$  is totally positive for every  $1 \leq n \leq M$ . Therefore Proposition 2.9 yields  $b_{n+1,n+1}^{n+1} \geq 0$ . So we see the lower estimate in (5.6) immediately by glancing at formula (5.4) for  $b_{n+1,n+1}^{n+1}$ . For the upper estimate, we obtain as a consequence of the inductive hypothesis

$$\begin{aligned} b_{n+1,n+1}^{n+1} &= 3 \left( (20)_{n+1} - \frac{1}{12} b_{n,n}^n (21)_n^2 \right)^{-1} \\ &\leq 3 \left( (20)_{n+1} - \frac{3}{12} \left( \frac{3}{4}(10)_n + (21)_n \right)^{-1} (21)_n^2 \right)^{-1} \\ &\leq 3 \left( (20)_{n+1} - \frac{1}{4} (21)_n \right)^{-1} \\ &= 3 \left( \frac{3}{4}(10)_{n+1} + (21)_{n+1} \right)^{-1}, \end{aligned}$$

thus the conclusion of the lemma.  $\square$

**Lemma 5.2.** *Let the matrix  $A$  be as above and  $1 \leq n \leq M$ . Then, the elements of the rightmost column of  $B_n$  satisfy the geometric estimate*

$$(5.7) \quad |b_{j,n}^n| \leq \frac{4q^{n-j}}{\eta(j,n)} \quad \text{for all } 1 \leq j \leq n,$$

where  $q = 2/3$ .

*Proof.* We infer from (5.4) and (5.5) that

$$b_{j,n+1}^{n+1} = -\frac{1}{6}b_{n+1,n+1}^{n+1}b_{j,n}^n(10)_{n+1}.$$

Invoking Lemma 5.1 for  $b_{n+1,n+1}^{n+1}$  we get further

$$(5.8) \quad |b_{j,n+1}^{n+1}| \leq \frac{2}{3} \frac{(10)_{n+1}}{(20)_{n+1}} |b_{j,n}^n|.$$

Now we note that by Lemma 5.1, inequality (5.7) is true for  $n = j$ . For general  $n > j$ , inequality (5.7) follows from (5.8) by induction using axiom (3) of  $\eta$  in Definition 2.5.  $\square$

**Theorem 5.3.** *Let the matrix  $A$  be as above and  $1 \leq n \leq M$ . Then, the elements of  $B_n = A_n^{-1}$  satisfy the geometric estimate*

$$|b_{i,j}^n| \leq \frac{36}{5} \frac{q^{|i-j|}}{\eta(i,j)} \quad \text{for all } 1 \leq i, j \leq n,$$

where  $q = 2/3$ .

*Proof.* We first observe that it is sufficient to prove the theorem for the parameter choices  $i \leq j \leq n-1$ , since  $B_n$  is symmetric and the case  $j = n$  is already covered by Lemma 5.2. Equations (5.4) and (5.5) yield

$$b_{i,j}^n = b_{i,j}^{n-1} + b_{n,n}^n b_{i,n-1}^{n-1} b_{j,n-1}^{n-1} \frac{(10)_n^2}{36},$$

so by induction

$$b_{i,j}^n = b_{i,j}^j + \sum_{l=j}^{n-1} b_{l+1,l+1}^{l+1} b_{i,l}^l b_{j,l}^l \frac{(10)_{l+1}^2}{36}.$$

Using now Lemmas 5.1 and 5.2 we get

$$\begin{aligned} |b_{i,j}^n| &\leq |b_{i,j}^j| + \frac{16}{9} \sum_{l=j}^{n-1} \frac{q^{2l-i-j} (10)_{l+1}^2}{\eta(i,l) \eta(j,l) (20)_{l+1}} \\ &\leq \frac{4q^{j-i}}{\eta(i,j)} \left( 1 + \frac{4}{9} \sum_{l=j}^{n-1} \frac{q^{2(l-j)} \eta(i,j) (10)_{l+1}^2}{\eta(i,l) \eta(j,l) (20)_{l+1}} \right). \end{aligned}$$

Since  $(10)_{l+1} \leq (20)_{l+1}$ ,  $(10)_{l+1} \leq (20)_l \leq \eta(j,l)$  and  $\eta(i,j) \leq \eta(i,l)$  for  $j \leq l \leq n-1$ , we estimate this from above by

$$\frac{4q^{j-i}}{\eta_{i,j}} \left( 1 + \frac{4}{9} \sum_{l=j}^{\infty} q^{2(l-j)} \right) = \frac{36}{5} \frac{q^{j-i}}{\eta(i,j)},$$

proving the theorem.  $\square$

**Remark 5.4.** An exponential estimate related to the one of Theorem 5.3 for the inverse Gramian  $B = (b_{i,j})_{i,j=1}^M$  of piecewise linear spline functions was first proved by [Cie66]. Another proof, using the B-splines  $N_i$  can be found in [KS89] yielding the estimate

$$|b_{i,j}| \leq 4 \frac{2^{-|i-j|}}{|\text{supp } N_i| \vee |\text{supp } N_j|}.$$

The same argument in fact yields

$$|b_{ij}| \leq 4 \frac{2^{-|i-j|}}{\eta_1(i,j)},$$

which is better than the assertion of Theorem 5.3. Nevertheless, this proof strongly uses the tridiagonal structure of the underlying matrix  $A$  and thus cannot be generalized to Gram matrices of higher order splines.

The reason why we presented Theorem 5.3 anyway, is that the basic structure of its proof is the same as in the more general cases of piecewise quadratic splines in Section 6.

## 6. PIECEWISE QUADRATIC SPLINES

In this section, we apply Corollary 4.1 to the case of piecewise quadratic splines to get geometric estimates for the entries of their Gram matrix inverse. We let  $\Delta = (s_i)_{i=1}^L$  be a partition of the unit interval  $[0, 1]$  as in (2.1) and set  $k = 3$ . The multiplicity vector  $\mu$  in the case for piecewise quadratic continuously differentiable splines is given by

$$\mu = (3, 1, 1, \dots, 1, 1, 3).$$

We construct the extended sequence of points  $\bar{\Delta} = \bar{\Delta}(\Delta, \mu) = (t_i)_{i=1}^{|\mu|}$  and the B-spline functions  $N_i \equiv N_{i,3}$  for  $1 \leq i \leq M = |\mu| - 3$  as in Section 2.2, that is

$$0 = t_1 = t_2 = t_3 < t_4 < \dots < t_{|\mu|-3} < t_{|\mu|-2} = t_{|\mu|-1} = t_{|\mu|} = 1.$$

**6.1. The Gram matrix.** We now calculate the Gram matrix of these B-splines. There is a standard formula for the inner product of B-splines of arbitrary order  $k$  involving divided differences (see for instance [Sch81], Theorem 4.25), but in this section we prefer to calculate the Gram matrix for  $k = 3$  directly. As an easy consequence of the recursion formula, point (4) of Proposition 2.4, we get the following

**Corollary 6.1.** *The B-spline functions  $N_i \equiv N_{i,3}$ ,  $1 \leq i \leq M$  of order 3 corresponding to the knot sequence  $\bar{\Delta} = (t_i)_{i=1}^{|\mu|}$  are given by*

$$(6.1) \quad N_i(x) = \begin{cases} \frac{(x - t_i)^2}{(20)_i(10)_i} & \text{if } x \in [t_i, t_{i+1}), \\ \frac{(x - t_i)(t_{i+2} - x)}{(20)_i(21)_i} + \frac{(x - t_{i+1})(t_{i+3} - x)}{(31)_i(21)_i} & \text{if } x \in [t_{i+1}, t_{i+2}), \\ \frac{(t_{i+3} - x)^2}{(31)_i(32)_i} & \text{if } x \in [t_{i+2}, t_{i+3}), \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 6.2.** *Let  $1 \leq i \leq M - 1$ . Then we have*

$$(6.2) \quad \int_{t_{i+1}}^{t_{i+2}} N_i(x) N_{i+1}(x) dx = \frac{(21)_i^2}{(31)_i} \left[ \frac{1}{10} + \frac{(10)_i}{30(20)_i} + \frac{(32)_i}{5(31)_i} \right]$$

and due to symmetry

$$(6.3) \quad \int_{t_{i+2}}^{t_{i+3}} N_i(x) N_{i+1}(x) dx = \frac{(32)_i^2}{(31)_i} \left[ \frac{1}{10} + \frac{(43)_i}{30(42)_i} + \frac{(21)_i}{5(31)_i} \right]$$

*Proof.* A straightforward calculation using Corollary 6.1 yields

$$\int_{t_{i+1}}^{t_{i+2}} N_i(x) N_{i+1}(x) dx = \frac{(21)_i^2}{(31)_i} \left[ \frac{(21)_i}{20(20)_i} + \frac{(10)_i}{12(20)_i} + \frac{1}{4} - \frac{(21)_i}{5(31)_i} \right].$$

An easy reformulation of this identity gives us (6.2). Equation (6.3) now follows by symmetry.  $\square$

**Proposition 6.3.** *We have the following formulas for the Gram matrix  $A = (a_{i,j})_{i,j=1}^M = (\langle N_i, N_j \rangle)_{i,j=1}^M$  of B-splines of order 3.*

$$(6.4) \quad a_{i,i+2} = a_{i+2,i} = \frac{(32)_i^3}{30(31)_i(42)_i} \quad \text{if } 1 \leq i \leq M - 2,$$

$$(6.5) \quad a_{i,i+1} = a_{i+1,i} = \frac{(31)_i}{10} + \frac{(21)_i^2(10)_i}{30(20)_i(31)_i} + \frac{(32)_i^2(43)_i}{30(31)_i(42)_i} \quad \text{if } 1 \leq i \leq M - 1,$$

$$(6.6) \quad a_{i,i} = \frac{(30)_i}{5} - \frac{(30)_i(21)_i^2}{15(20)_i(31)_i} \quad \text{if } 1 \leq i \leq M.$$

Furthermore,  $a_{i,j} = 0$  if  $|i - j| > 2$ .

*Proof.* Equation (6.4) is a simple consequence of Corollary 6.1. For equation (6.5), we observe that  $\langle N_i, N_{i+1} \rangle$  is the sum of (6.2) and (6.3). Thus, (6.5) is a consequence of the identity

$$\frac{(21)_i^2}{(31)_i} \left[ \frac{1}{10} + \frac{(32)_i}{5(31)_i} \right] + \frac{(32)_i^2}{(31)_i} \left[ \frac{1}{10} + \frac{(21)_i}{5(31)_i} \right] = \frac{(21)_i^2 + (32)_i^2}{10(31)_i} + \frac{(21)_i(32)_i}{5(31)_i} = \frac{(31)_i}{10},$$

where every bracket  $(mn)$  is taken with respect to the subindex  $i$ . We observe that from properties 3 and 5 of Proposition 2.4 we obtain

$$(6.7) \quad \left\langle N_i, \sum_{l=i-2}^{i+2} N_l \right\rangle = \langle N_i, 1 \rangle = \frac{(30)_i}{3}.$$

Since the only part in this equation that is unknown is  $\langle N_i, N_i \rangle$ , we use (6.7) and simple calculations to conclude the proof of the proposition.  $\square$

Observe that the Gram matrix  $A$  is symmetric, 2-banded and totally positive (see Remark 2.10), so in particular, we can apply Lemmas 4.2 and 4.4 to  $A$ .

**6.2. Continuous piecewise quadratic splines.** As a preliminary case, we examine the setting

$$(6.8) \quad \mu = (3, 2, 2, \dots, 2, 2, 3).$$

where every knot in the mesh sequence is a double knot. Thus, the sequence  $(t_i)_{i=1}^{|\mu|}$  has the form

$$(6.9) \quad 0 = t_1 = t_2 = t_3 < t_4 = t_5 < t_6 = t_7 < \dots$$

By Theorem 2.3, this corresponds to the space of piecewise quadratic splines that are continuous at the mesh points  $s_i, 1 \leq i \leq L$ . In fact, this is a special case of the choice  $\mu = (3, 1, 1, \dots, 1, 1, 3)$  since the Gram matrix  $A$  of the B-splines depends continuously on the point sequence  $(t_i)_{i=1}^{|\mu|}$  (see for instance [Sch81], Theorem 4.26). It follows from Proposition 6.3 and this continuity property that in the case of (6.8), formulas (6.4)–(6.6) for the Gram matrix  $A = (a_{i,j})_{i,j=1}^M$  simplify to

$$(6.10) \quad a_{i,i} = \langle N_i, N_i \rangle = \begin{cases} \frac{(30)_i}{5} & \text{if } i \text{ is odd} \\ \frac{2(21)_i}{15} & \text{if } i \text{ is even} \end{cases} \quad \text{if } 1 \leq i \leq M,$$

$$(6.11) \quad a_{i,i+1} = \langle N_i, N_{i+1} \rangle = \frac{(31)_i}{10} \quad \text{if } 1 \leq i \leq M-1,$$

$$(6.12) \quad a_{i,i+2} = \langle N_i, N_{i+2} \rangle = \frac{(32)_i}{30} \quad \text{if } 1 \leq i \leq M-2.$$

There are two reasons why present the special case of (6.8). First, we obtain better estimates than the general ones of Section 6.3. Secondly, in contrast to Section 6.3, we do all calculations by hand.

**Remark 6.4.** We note that (6.11) and (6.9) imply  $a_{n,n+2} = 0$  if  $n$  is even. Furthermore, if  $n$  is even, (6.10)–(6.12) yield the following relations between the entries of  $A$ .

$$(6.13) \quad 3a_{n-1,n+1} = a_{n,n+1}, \quad a_{n-1,n} = a_{n,n+1}, \quad 3a_{n,n} = 4a_{n-1,n}.$$

**Lemma 6.5.** If  $A = (a_{i,j})_{i,j=1}^M$  is defined as in (6.10)–(6.12) and  $2 \leq n \leq M$  is even, we have

$$(6.14) \quad b_{j,n-1}^n = \frac{\delta_{j,n} - b_{j,n}^n a_{n,n}}{a_{n-1,n}} = \frac{\delta_{j,n}}{a_{n-1,n}} - \frac{4}{3} b_{j,n}^n \quad \text{for } 1 \leq j \leq n.$$

*Proof.* If  $n$  is even, the  $n$ -th row of the matrix  $A$  has only three non-zero entries by Remark 6.4. So the  $n$ -th row of  $A_n$  has only two non-zero entries and thus we have by the defining property of  $B_n = (A_n)^{-1}$

$$\delta_{j,n} = \sum_{i=1}^n b_{j,i}^n a_{i,n} = \sum_{i=n-1}^n b_{j,i}^n a_{i,n} = b_{j,n-1}^n a_{n-1,n} + b_{j,n}^n a_{n,n} \quad \text{for every } 1 \leq j \leq n.$$

The assertion of the lemma now follows from these equations and (6.13).  $\square$

**Lemma 6.6.** If  $A = (a_{i,j})_{i,j=1}^M$  is defined as in (6.10)–(6.12) and  $2 \leq n \leq M$  is even, we have

$$(6.15) \quad b_{n,n}^n = (a_{n,n} - b_{n-1,n-1}^{n-1} a_{n-1,n}^2)^{-1},$$

$$(6.16) \quad b_{n+1,n+1}^{n+1} = \left( a_{n+1,n+1} - \frac{25}{81} b_{n,n}^n a_{n,n+1}^2 - \frac{14}{27} a_{n,n+1} \right)^{-1}.$$

*Proof.* The first equation (6.15) is a consequence of (4.4) and the fact that the  $n$ -th row of  $A_n$  has only two non-zero entries. For the second equation we observe that by (4.4) and the checkerboard property of  $B_n = (A_n)^{-1}$  (see Remark 2.10)

$$b_{n+1,n+1}^{n+1} = (a_{n+1,n+1} - b_{n,n}^n a_{n,n+1}^2 + 2|b_{n-1,n}^n| a_{n,n+1} a_{n-1,n+1} - b_{n-1,n-1}^n a_{n-1,n+1}^2)^{-1}.$$

Using Lemma 6.5 for  $b_{n-1,n}^n$  and  $b_{n-1,n-1}^n$  we obtain

$$b_{n+1,n+1}^{n+1} = \left( a_{n+1,n+1} - b_{n,n}^n \left[ a_{n,n+1}^2 - \frac{8}{3} a_{n,n+1} a_{n-1,n+1} + \frac{16}{9} a_{n-1,n+1}^2 \right] - 2 \frac{a_{n,n+1} a_{n-1,n+1}}{a_{n-1,n}} + \frac{4}{3} \frac{a_{n-1,n+1}^2}{a_{n-1,n}} \right)^{-1}.$$

By (6.13), this becomes

$$b_{n+1,n+1}^{n+1} = (a_{n+1,n+1} - \frac{25}{81} b_{n,n}^n a_{n,n+1}^2 - \frac{14}{27} a_{n,n+1})^{-1},$$

and thus the lemma is proved.  $\square$

**Proposition 6.7.** *If  $A = (a_{i,j})_{i,j=1}^M$  is defined as in (6.10)–(6.12) and  $1 \leq n \leq M$ , we have the estimate*

$$(6.17) \quad b_{n,n}^n \leq \left( \frac{(10)_n}{9} + \frac{(21)_n}{12} + \frac{(32)_n}{5} \right)^{-1}.$$

*Proof.* By induction. If  $n = 1$

$$b_{1,1}^1 = a_{1,1}^{-1} = \frac{5}{(30)_1} = \left( \frac{(10)_1}{9} + \frac{(21)_1}{12} + \frac{(32)_1}{5} \right)^{-1},$$

since  $(10)_1 = (21)_1 = 0$ .

In the first place, we assume that  $n$  is even. In this case we get by equation (6.15) of Lemma 6.6, (6.10)–(6.12) and the inductive hypothesis respectively

$$\begin{aligned} b_{n,n}^n &= (a_{n,n} - b_{n-1,n-1}^{n-1} a_{n-1,n}^2)^{-1} \\ &= \left( \frac{2(21)_n}{15} - b_{n-1,n-1}^{n-1} \left( \frac{(21)_n}{10} \right)^2 \right)^{-1} \\ &\leq \left( \frac{2(21)_n}{15} - \left( \frac{(10)_{n-1}}{9} + \frac{(32)_{n-1}}{5} \right)^{-1} \left( \frac{(21)_n}{10} \right)^2 \right)^{-1} \\ &\leq \left( \frac{2(21)_n}{15} - \frac{(21)_n}{20} \right)^{-1} = \frac{12}{(21)_n} = \left( \frac{(10)_n}{9} + \frac{(21)_n}{12} + \frac{(32)_n}{5} \right)^{-1}, \end{aligned}$$

where we used that  $(10)_n = (32)_n = 0$ . Thus the claimed inequality (6.17) is proved for  $n$  even.

Secondly, assume that  $n$  is odd. Here we get by equation (6.16) of Lemma 6.6, (6.10)–(6.12) and the inductive hypothesis respectively

$$b_{n,n}^n = \left( a_{n,n} - \frac{25}{81} b_{n-1,n-1}^{n-1} a_{n-1,n}^2 - \frac{14}{27} a_{n-1,n} \right)^{-1}$$

$$\begin{aligned}
&= \left( \frac{(30)_n}{5} - \frac{25}{81} b_{n-1,n-1}^{n-1} \left( \frac{(21)_{n-1}}{10} \right)^2 - \frac{14}{27} \frac{(21)_{n-1}}{10} \right)^{-1} \\
&\leq \left( \frac{(30)_n}{5} - \frac{25}{81} \frac{12}{(30)_{n-1}} \left( \frac{(21)_{n-1}}{10} \right)^2 - \frac{14}{27} \frac{(21)_{n-1}}{10} \right)^{-1} \\
&= \left( \frac{(30)_n}{5} - \frac{4(10)_n}{45} \right)^{-1} = \left( \frac{(10)_n}{9} + \frac{(32)_n}{5} \right)^{-1} \\
&= \left( \frac{(10)_n}{9} + \frac{(21)_n}{12} + \frac{(32)_n}{5} \right)^{-1},
\end{aligned}$$

where we used that  $(21)_n = 0$ . This proves inequality (6.17) if  $n$  is odd. We thus proved the assertion of the proposition.  $\square$

**Lemma 6.8.** *If  $A = (a_{i,j})_{i,j=1}^M$  is defined as in (6.10)–(6.12) and  $2 \leq n \leq M$ , we have*

$$(6.18) \quad b_{j,n}^n = -b_{n,n}^n b_{j,n-1}^{n-1} a_{n-1,n} \quad \text{for every } 1 \leq j \leq n-1$$

if  $n$  is even and

$$(6.19) \quad b_{j,n}^n = -b_{n,n}^n \left( \frac{\delta_{j,n-1}}{3} + \frac{5}{9} b_{j,n-1}^{n-1} a_{n-1,n} \right) \quad \text{for every } 1 \leq j \leq n-1$$

if  $n$  is odd.

*Proof.* Equations (4.3)–(4.5) yield the formula

$$(6.20) \quad b_{j,n}^n = -b_{n,n}^n (b_{j,n-1}^{n-1} a_{n-1,n} + b_{j,n-2}^{n-1} a_{n-2,n}).$$

Since for  $n$  even,  $a_{n-2,n} = 0$ , equation (6.18) follows.

On the other hand, we get from Lemma 6.5 that  $b_{j,n-2}^{n-1} = \delta_{j,n-1}/a_{n-2,n-1} - 4b_{j,n-1}^{n-1}/3$  for  $1 \leq j \leq n-1$ . Combining this with (6.20), we obtain

$$b_{j,n}^n = -b_{n,n}^n b_{j,n-1}^{n-1} \left( a_{n-1,n} - \frac{4}{3} a_{n-2,n} \right) - \delta_{j,n-1} b_{n,n}^n \frac{a_{n-2,n}}{a_{n-2,n-1}}.$$

Using (6.13) in this equation, we obtain (6.19). This completes the proof of the lemma.  $\square$

**Proposition 6.9.** *If  $A = (a_{i,j})_{i,j=1}^M$  is defined as in (6.10)–(6.12) and  $2 \leq n \leq M$ , then the entries  $b_{j,n}^n$  of the last column of the matrix  $B_n = A_n^{-1}$  admit the geometric estimate*

$$(6.21) \quad |b_{j,n}^n| \leq C \frac{q^{n-j}}{\eta(j,n)} \quad \text{for every } 1 \leq j \leq n,$$

where  $q = (3/5)^{1/2}$  and  $C = 18$ .

*Proof.* We can use Lemma 6.8, Proposition 6.7 and (6.11) to show that

$$(6.22) \quad |b_{j,n}^n| \leq C_{j,n} \frac{q^{n-j}}{\eta(j,n)} \quad \text{for } n-2 \leq j \leq n,$$

where  $C_{j,n}$  satisfies the inequality  $C_{j,n} \leq 18$  for all  $n-2 \leq j \leq n$ . Indeed, for  $j = n$  this is a direct consequence of Proposition 6.7. If  $j = n-1$ , we use Lemma 6.8 to write  $b_{j,n}^n$  as an expression involving only  $b_{n,n}^n$ ,  $b_{n-1,n-1}^{n-1}$  and  $a_{n-1,n}$ . Then we are able to invoke again Proposition 6.7 and equation (6.11) to show (6.22) for



$j = n - 1$ . If  $j = n - 2$  we apply Lemma 6.8 to write  $b_{j,n}^n$  as an expression involving only  $b_{n,n}^n$ ,  $b_{n-2,n-1}^{n-1}$  and  $a_{n-1,n}$ . Here we use Proposition 6.7 for  $b_{n,n}^n$ , equation (6.11) for  $a_{n-1,n}$  and the already proved estimate (6.22) for  $b_{n-2,n-1}^{n-1}$ .

If  $j \leq n - 3$ , we obtain from Lemma 6.8 and (6.11)

$$\begin{aligned} |b_{j,n}^n| &= \frac{5}{9} b_{n-1,n-1}^{n-1} b_{n,n}^n a_{n-2,n-1} a_{n-1,n} |b_{j,n-2}^{n-2}| \\ &\leq \frac{5}{9} \left( \frac{(10)_{n-1}}{9} + \frac{(21)_{n-1}}{12} + \frac{(32)_{n-1}}{5} \right)^{-1} \left( \frac{(10)_n}{9} + \frac{(21)_n}{12} + \frac{(32)_n}{5} \right)^{-1} \\ &\quad \cdot \frac{(31)_{n-2}(31)_{n-1}}{100} |b_{j,n-2}^{n-2}| \\ &\leq \frac{5}{9} \frac{12 \cdot 9}{100} \frac{(31)_{n-2}(31)_{n-1}}{(30)_{n-1}(30)_n} |b_{j,n-2}^{n-2}|, \end{aligned}$$

since at least one of the two terms  $(21)_{n-1}$ ,  $(21)_n$  vanishes. The elementary inequality (cf. Axiom (3) in Definition 2.5)

$$\frac{(31)_{n-2}(31)_{n-1}}{(30)_{n-1}(30)_n \eta(j, n-2)} \leq \frac{1}{\eta(j, n)},$$

induction and (6.22) now yield the assertion of the proposition.  $\square$

Given Proposition 6.9, we basically use the same proof as the one of Theorem 5.3 to deduce the geometric decay off the diagonal for the inverse Gramian in the current case. This is formulated in the following

**Theorem 6.10.** *If  $A = (a_{i,j})_{i,j=1}^M$  is defined as in (6.10)–(6.12) and  $1 \leq n \leq M$ , then the entries  $b_{i,j}^n$  of the matrix  $B_n$  satisfy the geometric inequality*

$$(6.23) \quad |b_{i,j}^n| \leq C \left( 3 + \frac{3C}{10} \right) \frac{q^{|j-i|}}{\eta(i, j)} \quad \text{for } 1 \leq i, j \leq n,$$

where, as in Proposition 6.9,  $q = (3/5)^{1/2}$  and  $C = 18$ .

*Proof.* Since  $B_n$  is symmetric and the case  $j = n$  was treated in Proposition 6.9, it suffices to consider the case  $i \leq j \leq n - 1$ . At first we only look at the parameter choices  $i < j$  and  $j$  odd. Equations (4.3)–(4.5) yield the formula

$$(6.24) \quad b_{i,j}^n = b_{i,j}^{n-1} + b_{i,n}^n b_{j,n}^n / b_{n,n}^n.$$

Equation (6.24) and Lemma 6.8 imply with  $\alpha_n = 1$  if  $n$  is even and  $\alpha_n = 5/9$  if  $n$  is odd

$$(6.25) \quad b_{i,j}^n = b_{i,j}^{n-1} + \alpha_n^2 b_{n,n}^n b_{i,n-1}^{n-1} b_{j,n-1}^{n-1} a_{n-1,n}^2.$$

By induction,

$$(6.26) \quad b_{i,j}^n = b_{i,j}^j + \sum_{l=j}^{n-1} \alpha_{l+1}^2 b_{l+1,l+1}^{l+1} b_{i,l}^l b_{j,l}^l a_{l,l+1}^2.$$

We now use that  $\alpha_l \leq 1$  for all  $l$  and employ Propositions 6.9 and 6.7 to deduce from (6.26)

$$|b_{i,j}^n| \leq |b_{i,j}^j| + \sum_{l=j}^{n-1} b_{l+1,l+1}^{l+1} |b_{i,l}^l b_{j,l}^l| a_{l,l+1}^2$$

FIGURE 1: Definition of the Gram matrix  $A$  in MATHEMATICA.

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```

a[i_, j_] :=
  
$$\frac{\sum_{n=i}^{i+2} x[n]}{15} (3 - x[i+1]^2 / ((x[i] + x[i+1]) (x[i+1] + x[i+2]))) /; j == i;$$

a[i_, j_] := 
$$\frac{x[i+1] + x[i+2]}{10} +$$

  
$$\frac{(x[i+1]^2 x[i]) / (30 (x[i] + x[i+1]) (x[i+1] + x[i+2])) +$$

  
$$(x[i+2]^2 x[i+3]) / (30 (x[i+1] + x[i+2]) (x[i+2] + x[i+3]))}{10} /; j == i+1;$$

a[i_, j_] := 
$$x[i+2]^3 / (30 (x[i+1] + x[i+2]) (x[i+2] + x[i+3])) /; j == i+2;$$


```

---

FIGURE 2: Definition of the function  $\varphi$  in MATHEMATICA.

---

```

φ[i_] :=
  
$$\left( \frac{1}{9} x[i] + \frac{1}{12} x[i+1] + \frac{1}{5} x[i+2] - (x[i+1] x[i+2]) / (30 (x[i+1] + x[i+2])) \right.$$

  
$$+ (1 + x[i+2] / (6 (x[i+1] + x[i+2])) - (20 x[i]) / (9 (x[i] + x[i+1])))$$

  
$$+ (5 x[i-1] x[i]) / (108 (x[i-1] + x[i])) +$$

  
$$\left. (2 x[i-1]^2 x[i]) / (73 (x[i-1] + x[i])^2) \right)^{-1};$$


```

---

$$\begin{aligned}
&\leq C \frac{q^{j-i}}{\eta(i, j)} + \sum_{l=j}^{n-1} \frac{12}{(30)_{l+1}} C \frac{q^{l-i}}{\eta(i, l)} C \frac{q^{l-j}}{\eta(j, l)} \frac{(31)_l^2}{100} \\
&= C \frac{q^{j-i}}{\eta(i, j)} \left( 1 + \frac{3C}{25} \sum_{l=j}^{n-1} \frac{\eta(i, j) q^{2l-2j}}{\eta(i, l) \eta(j, l)} \frac{(31)_l^2}{(30)_{l+1}} \right),
\end{aligned}$$

where  $q, C$  are as in Proposition 6.9. Since  $\eta(i, j) \leq \eta(i, l)$ ,  $(31)_l \leq \eta(j, l)$  and  $(31)_l \leq (30)_{l+1}$  for all  $j \leq l \leq n-1$ , we get further

$$(6.27) \quad |b_{i,j}^n| \leq C \frac{q^{j-i}}{\eta(i, j)} \left( 1 + \frac{3C}{25} \sum_{l=j}^{n-1} q^{2(l-j)} \right) \leq C \left( 1 + \frac{3C}{10} \right) \frac{q^{j-i}}{\eta(i, j)}.$$

If  $j = i$  or  $j$  is even, the proof follows the same lines, but—in view of Lemma 6.8—with a bit more notation, so we omit it. This is the place where the summand 3 appears in (6.23) instead of 1 in the right hand side of (6.27).  $\square$

**6.3. Continuously differentiable piecewise quadratic splines.** In this section we come back to the general case for piecewise quadratic splines, where  $k = 3$ ,  $\mu = (3, 1, 1, \dots, 1, 1, 3)$  and the Gram matrix  $A$  of the splines  $N_i \equiv N_{i,3}$  has the form (6.4)–(6.6). In the following, we use a computer algebra system, in our case MATHEMATICA 8.0, to show that certain given polynomials have only nonnegative coefficients. This allows us to deduce that the given polynomial itself is nonnegative for positive arguments. In the following results, we need the matrix  $A$  to be defined in MATHEMATICA. This is done in Figure 1. Furthermore we need a MATHEMATICA expression for  $\varphi_n$  to be defined in Proposition 6.11. For this, we refer to Figure 2.

The first thing to show is, as in Section 6.2, a suitable upper bound for  $b_{nn}^n$ . This is the content of the following Proposition 6.11.

**Proposition 6.11.** *Let  $A$  be as in Proposition 6.3 and  $1 \leq n \leq M$ . Then the bottommost rightmost element  $b_{n,n}^n$  of  $B_n = A_n^{-1}$  satisfies the estimate*

$$(6.28) \quad b_{n,n}^n \leq \left( \frac{(10)}{9} + \frac{(21)}{12} + \frac{(32)}{5} - \frac{(21)(32)}{30(31)} \left( 1 + \frac{(32)}{6(31)} - \frac{20(10)}{9(20)} \right) + \frac{5(0,-1)(10)}{108(1,-1)} + \frac{2(0,-1)^2(10)}{73(1,-1)^2} \right)^{-1} =: \varphi_n,$$

where every bracket  $(ij)$  in this formula is taken with respect to index  $n$ .

**Remark 6.12.** Observe that in the limiting case  $(10)_n, (32)_n \rightarrow 0$  for  $n$  even,  $\varphi_n$  is the estimate of Proposition 6.7.

*Proof.* We use Lemma 4.2 and induction to prove the claimed estimate. Recall that Lemma 4.2 stated that

$$(6.29) \quad b_{n+1,n+1}^{n+1} \leq \left( a_{n+1,n+1} - b_{n,n}^n a_{n,n+1} \left( a_{n,n+1} - \frac{2a_{n,n}a_{n-1,n+1}}{a_{n-1,n}} \right) - 2 \frac{a_{n,n+1}a_{n-1,n+1}}{a_{n-1,n}} - a_{n-1,n+1}^2 b_{n-1,n-1}^{n-1} (1 + b_{n,n}^n b_{n-1,n-1}^{n-1} a_{n-1,n}^2) \right)^{-1}$$

for  $2 \leq n \leq M-1$ . Thus to apply induction, we need to verify (6.28) for  $n=1$  and  $n=2$ . First, we suppose that  $n=1$ ; here we have  $b_{1,1}^1 = a_{1,1}^{-1} = 5/(30)_1$  by (6.6) and the fact that  $(20)_1 = 0$ . For  $n=2$ , the formula  $b_{2,2}^2 = (a_{2,2} - b_{1,1}^1 a_{1,2}^2)^{-1}$  holds in view of Corollary 4.1. We thus obtain by (6.5), (6.6),  $(20)_1 = 0$  and some elementary calculations

$$(6.30) \quad b_{2,2}^2 = \left( \frac{(21)_2}{12} + \frac{(32)_2}{5} - \frac{(21)_2(32)_2}{30(31)_2} \left( 1 + \frac{(32)_2}{6(31)_2} \right) \right)^{-1}.$$

The expression for  $b_{1,1}^1$  and (6.30) are special cases of (6.28) since  $(20)_1 = 0$ . Thus, the lemma is proved for  $n=1$  and  $n=2$ .

Before we proceed with the actual proof of (6.28), we show that the term  $a_{n,n+1} - \frac{2a_{n,n}a_{n-1,n+1}}{a_{n-1,n}}$ , appearing in (6.29), is nonnegative. It is equivalent to show that  $a_{n,n+1}a_{n-1,n} - 2a_{n,n}a_{n-1,n+1}$  is nonnegative. This is done using the MATHEMATICA-code of Figure 3. The method of proof is as follows.

- (1) Define the rational function  $p = a_{n,n+1}a_{n-1,n} - 2a_{n,n}a_{n-1,n+1}$  depending on the variables  $x[1] = (10)_{n-1}, x[2] = (21)_{n-1}, \dots, x[5] = (54)_{n-1}$ .
- (2) Determine the denominator  $d$  of the rational function  $p$  as  $900(x[1] + x[2])(x[2] + x[3])^2(x[3] + x[4])^2(x[4] + x[5])$  and observe that  $d$  is positive for positive arguments.
- (3) Calculate the coefficients of the polynomial  $q := d \cdot p$  and verify that no coefficient of  $q$  is negative.

Now we continue with the proof of (6.28). Since every entry  $a_{ij}$  of the Gram matrix  $A$  is nonnegative and the matrices  $B_{n-1}, B_n$  are checkerboard, the argument of the last paragraph shows that a sufficient condition for (6.28) to be true

FIGURE 3: Proof of the inequality  $a_{n,n+1} - 2a_{nn}a_{n-1,n+1}/a_{n-1,n} \geq 0$ .

---

```

In[11]:= p[z_] := Factor[a[1, 2] a[2, 3] - 2 a[2, 2] a[1, 3] /.
      Table[x[n] -> List[z][[n]], {n, 1, Length[List[z]]}]]];
d[z_] := Denominator[p[z]];
q[z_] := d[z] p[z];
d[Sequence @@ Table[x[n], {n, 1, 5}]]
Select[Flatten[CoefficientList[
      q[Sequence @@ Table[x[n], {n, 1, 5}]], Table[x[n], {n, 1, 5}]]], # < 0 &]

Out[14]= 900 (x[1] + x[2]) (x[2] + x[3])^2 (x[3] + x[4])^2 (x[4] + x[5])

Out[15]= {}

```

---

for all  $1 \leq n \leq M$  is the following recursive inequality for  $\varphi_n$ ,  $2 \leq n \leq M - 1$ .

$$(6.31) \quad \left( a_{n+1,n+1} - \varphi_n a_{n,n+1} \left( a_{n,n+1} - \frac{2a_{n,n}a_{n-1,n+1}}{a_{n-1,n}} \right) - 2 \frac{a_{n,n+1}a_{n-1,n+1}}{a_{n-1,n}} - a_{n-1,n+1}^2 \varphi_{n-1} (1 + \varphi_n \varphi_{n-1} a_{n-1,n}^2) \right)^{-1} \leq \varphi_{n+1}.$$

The proof that this inequality is true for all choices of nonnegative distances  $(0, -1)_{n-1}, (10)_{n-1}, \dots, (54)_{n-1}$  is equivalent to prove that the rational function  $p$  defined as

$$(6.32) \quad \varphi_n^{-1} \varphi_{n-1}^{-1} a_{n-1,n} \left( \left( a_{n+1,n+1} - \varphi_n a_{n,n+1} \left( a_{n,n+1} - \frac{2a_{n,n}a_{n-1,n+1}}{a_{n-1,n}} \right) - 2 \frac{a_{n,n+1}a_{n-1,n+1}}{a_{n-1,n}} - a_{n-1,n+1}^2 \varphi_{n-1} (1 + \varphi_n \varphi_{n-1} a_{n-1,n}^2) \right) - \varphi_{n+1}^{-1} \right)$$

and depending on the six variables  $(0, -1)_{n-1}, (10)_{n-1}, \dots, (54)_{n-1}$  is nonnegative for nonnegative arguments. This is done in the MATHEMATICA-code of Figure 4 using the following steps.

- (1) Define the rational function  $p$  to be the expression in (6.32) depending on the variables  $x[1] = (10)_{n-2}, x[2] = (21)_{n-2}, \dots, x[5] = (54)_{n-2}, x[6] = (65)_{n-2}$ .
- (2) Determine the denominator  $d$  of the rational function  $p$  as an integer multiple of  $(x[1] + x[2])^4 (x[2] + x[3])^5 (x[3] + x[4])^8 (x[4] + x[5])^5 (x[5] + x[6])^2$  and observe that  $d$  is positive for positive arguments.
- (3) Calculate the coefficients of the polynomial  $q := d \cdot p$  and verify that no coefficient of  $q$  is negative.

This proves the assertion of the proposition □

**Remark 6.13.** It is an easy consequence of the above lemma and the inequality

$$\frac{(21)(32)}{(31)} \left( 1 + \frac{(32)}{6(31)} \right) \leq (32),$$

that we also have the estimate

$$(6.33) \quad b_{n,n}^n \leq \left( \frac{(10)}{9} + \frac{(21)}{12} + \frac{(32)}{6} \right)^{-1} =: \psi_n$$

for all  $1 \leq n \leq M$ . The MATHEMATICA expression of  $\psi_n$  is given by Figure 5.

FIGURE 4: Proof of the estimate for  $b_{nn}^n$ .

---

```

In[31]:= p[z___] := Factor[φ[3]-1 φ[2]-2 a[2, 3]
      (a[4, 4] - φ[3] a[3, 4] (a[3, 4] - (2 a[3, 3] a[2, 4]) / a[2, 3]) -
      
$$\frac{2 a[3, 4] a[2, 4]}{a[2, 3]} - a[2, 4]^2 φ[2] (1 + φ[3] φ[2] a[2, 3]^2) - φ[4]^{-1} \Big) /.
      Table[x[n] → List[z][[n]], {n, 1, Length[List[z]]}]];
d[z___] := Denominator[p[z]];
q[z___] := d[z] p[z];
d[Sequence @@ Table[x[n], {n, 1, 6}]]
Select[Flatten[CoefficientList[
      q[Sequence @@ Table[x[n], {n, 1, 6}]], Table[x[n], {n, 1, 6}]]], # < 0 &]
Out[34]= 72 441 550 057 348 800 000 (x[1] + x[2])4
      (x[2] + x[3])5 (x[3] + x[4])8 (x[4] + x[5])5 (x[5] + x[6])2
Out[35]= {}$$

```

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FIGURE 5: Definition of the function  $\psi$ .

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$$\psi[i\_ ] := \left( \frac{1}{9} x[i] + \frac{1}{12} x[i+1] + \frac{1}{6} x[i+2] \right)^{-1};$$


---

**Lemma 6.14.** *Let  $A$  be as in Proposition 6.3 and  $2 \leq n \leq M$ . Then we have the estimate*

$$b_{n,n}^n a_{n-1,n} \leq \frac{6}{5} \frac{(20)_n}{(30)_n}.$$

*Proof.* By Remark 6.13, it suffices to show that

$$\psi_n a_{n-1,n} \leq \frac{6}{5} \frac{(20)_n}{(30)_n}.$$

We apply the same method of proof as in the above proposition and proceed with the MATHEMATICA-code of Figure 6 using the following steps.

- (1) Define the rational function  $p = \frac{6}{5} \frac{(20)_n}{(30)_n} - \psi_n a_{n-1,n}$  depending on the variables  $x[1] = (10)_{n-1}$ ,  $x[2] = (21)_{n-1}$ ,  $x[3] = (32)_{n-1}$ ,  $x[4] = (43)_{n-1}$ .
- (2) Determine the denominator  $d$  of the rational function  $p$  as  $5(x[1] + x[2])(x[2] + x[3])(x[3] + x[4])(x[2] + x[3] + x[4])(4x[2] + 3x[3] + 6x[4])$  and observe that  $d$  is positive for positive arguments.
- (3) Calculate the coefficients of the polynomial  $q := d \cdot p$  and verify that no coefficient of  $q$  is negative.  $\square$

Let  $\theta_n := b_{n,n}^n \left( a_{n-1,n} - \frac{a_{n-2,n} a_{n-1,n-1}}{a_{n-2,n-1}} \right)$ . This expression is important, since it is used in Lemma 4.4 to estimate  $|b_{j,n}^n|$  for  $1 \leq j \leq n-1$ . In the following lemma, we estimate the product of two consecutive values of  $\theta_n$ . We will use this result in the proof of Proposition 6.16 to obtain explicit estimates for  $|b_{j,n}^n|$ .

FIGURE 6: Proof of the estimate for  $b_{n,n}^n a_{n-1,n}$ .

---

```

In[21]:= p[z_] := Factor[ $\frac{6}{5} \frac{x[2] + x[3]}{x[2] + x[3] + x[4]} - \psi[2] a[1, 2]$ ] /.
      Table[x[j] → List[z][[n]], {n, 1, Length[List[z]]}]];
d[z_] := Denominator[p[z]];
q[z_] := d[z] p[z];
d[Sequence @@ Table[x[n], {n, 1, 4}]]
Select[Flatten[CoefficientList[q[x, y, z, w], {x, y, z, w}], # < 0 &]

Out[24]= 5 (x[1] + x[2]) (x[2] + x[3]) (x[3] + x[4])
      (x[2] + x[3] + x[4]) (4 x[2] + 3 x[3] + 6 x[4])

Out[25]= {}

```

---

FIGURE 7: Proof of the estimate for the product of two consecutive values of  $\theta_n$ .

---

```

In[26]:= p[z_] := Factor[ $\frac{87}{100} - \varphi[3] \varphi[4] \frac{x[3] + x[4] + x[5]}{x[3] + x[4]} \frac{x[4] + x[5] + x[6]}{x[4] + x[5]}$ 
       $\left( a[2, 3] - \frac{a[1, 3] a[2, 2]}{a[1, 2]} \right) \left( a[3, 4] - \frac{a[2, 4] a[3, 3]}{a[2, 3]} \right) /.$ 
      Table[x[n] → List[z][[n]], {n, 1, Length[List[z]]}]];
d[z_] := Denominator[p[z]];
q[z_] := d[z] p[z];
Select[Flatten[CoefficientList[
      d[Sequence @@ Table[x[n], {n, 1, 6}]], Table[x[n], {n, 1, 6}]]], # < 0 &]
Select[Flatten[CoefficientList[q[Sequence @@ Table[x[n], {n, 1, 6}]],
      Table[x[n], {n, 1, 6}]]], # < 0 &]

Out[29]= {}

Out[30]= {}

```

---

**Lemma 6.15.** *Let  $A$  be as in Proposition 6.3 and  $3 \leq n \leq M - 1$ . Then we have that*

$$(6.34) \quad \theta_n \theta_{n+1} \leq \frac{87}{100} \frac{(20)_n (20)_{n+1}}{(30)_n (30)_{n+1}}.$$

*Proof.* We show that the rational function  $p$ , defined as

$$(6.35) \quad \frac{87}{100} - \frac{(30)_n (30)_{n+1}}{(20)_n (20)_{n+1}} \theta_n \theta_{n+1},$$

depending on the variables  $x[1] = (10)_{n-2}, x[2] = (21)_{n-2}, \dots, x[6] = (65)_{n-2}$  is nonnegative for nonnegative arguments. This is done with the MATHEMATICA-code in Figure 7 using the steps

- (1) Define the rational function  $p$  as in (6.35).
- (2) Determine the denominator  $d$  of the rational function  $p$  and verify that no coefficient of the polynomial  $d$  is negative.
- (3) Determine the coefficients of the polynomial  $q := d \cdot p$  and verify that no coefficient of  $q$  is negative.  $\square$

**Proposition 6.16.** *Let  $A$  be as in Proposition 6.3,  $1 \leq n \leq M$  and  $1 \leq j \leq n$ . Then we have the estimate*

$$|b_{j,n}^n| \leq C \frac{q^{n-j}}{\eta(j,n)} \quad \text{with } q = (87/100)^{1/2}, \quad C = 12q^{-2} \left(\frac{6}{5}\right)^2.$$

*Proof.* We see in the first place that Remark 6.13 yields the estimate  $b_{n,n}^n \leq 12/(30)_n$  and thus the assertion of the theorem in the case  $j = n$ . If  $j \leq n - 1$ , we get from Lemma 4.4

$$(6.36) \quad |b_{j,n}^n| \leq b_{j,j}^j b_{j+1,j+1}^{j+1} a_{j,j+1} \prod_{l=j+2}^n \theta_l.$$

Now assume that  $n - (j + 2) + 1 = n - j - 1$  is odd. By Lemmas 6.14 and 6.15 we conclude from (6.36) that

$$(6.37) \quad |b_{j,n}^n| \leq \frac{12}{(30)_j} \left(\frac{6}{5}\right)^2 q^{n-j-2} \prod_{l=j+1}^n \frac{(20)_l}{(30)_l}.$$

We apply axiom (3) of Definition 2.5 on  $\eta$  and induction to obtain the final estimate

$$(6.38) \quad |b_{j,n}^n| \leq 12q^{-2} \left(\frac{6}{5}\right)^2 \frac{q^{n-j}}{\eta(j,n)}.$$

If we assume that  $n - (j + 2) + 1$  is even, the same proof yields the estimate

$$(6.39) \quad |b_{j,n}^n| \leq 12q^{-1} \frac{6}{5} \frac{q^{n-j}}{\eta(j,n)}.$$

Inequalities (6.38) and (6.39) together now prove the proposition.  $\square$

The passage to estimates of expressions  $|b_{i,j}^n|$  for  $1 \leq i, j \leq n$  is now an analogous calculation as in Theorem 5.3 and carried out in the following

**Theorem 6.17.** *Let  $A$  be as in Proposition 6.3 and  $1 \leq n \leq M$ . Then the entries  $b_{i,j}^n$  of the matrix  $B_n$  satisfy the estimate*

$$(6.40) \quad |b_{i,j}^n| \leq C_1 \frac{q^{|i-j|}}{\eta(i,j)} \quad \text{for all } 1 \leq i, j \leq n,$$

where  $C_1 = C(1 + \frac{12}{75} \frac{C}{1-q^2})$  and  $C, q$  are as in Proposition 6.16.

*Proof.* Since  $B_n$  is symmetric and the case  $j = n$  was treated in Proposition 6.16, it suffices to consider the cases  $i \leq j \leq n - 1$ . Equations (4.3) – (4.5) yield the formula

$$b_{i,j}^n = b_{i,j}^{n-1} + b_{i,n}^n b_{j,n}^n / b_{n,n}^n.$$

By Lemma 4.4, inequality (4.10), we obtain

$$|b_{i,j}^n| \leq |b_{i,j}^{n-1}| + b_{n,n}^n a_{n-1,n}^2 |b_{i,n-1}^{n-1} b_{j,n-1}^{n-1}|.$$

Applying Lemma 6.14 and Proposition 6.16 we estimate further

$$|b_{i,j}^n| \leq |b_{i,j}^{n-1}| + C^2 \frac{6}{5} a_{n-1,n} \frac{(20)_n}{(30)_n} \frac{q^{2n-2-i-j}}{\eta(i,n-1)\eta(j,n-1)}.$$

By induction, we get

$$\begin{aligned} |b_{i,j}^n| &\leq |b_{i,j}^j| + \frac{6}{5}C^2 \sum_{l=j}^{n-1} a_{l,l+1} \frac{(20)_{l+1}}{(30)_{l+1}} \frac{q^{2l-i-j}}{\eta(i,l)\eta(j,l)} \\ &\leq C \frac{q^{j-i}}{\eta(i,j)} \left( 1 + \frac{6}{5}C \sum_{l=j}^{n-1} \frac{(20)_{l+1}}{(30)_{l+1}} a_{l,l+1} \frac{q^{2(l-j)}\eta(i,j)}{\eta(i,l)\eta(j,l)} \right). \end{aligned}$$

An easy consequence of formula (6.5) for  $a_{l,l+1}$  is that  $a_{l,l+1} \leq 2(31)_l/15 \leq 2\eta(i,l)/15$ . This and the obvious inequalities  $\eta(i,j) \leq \eta(i,l)$ ,  $(20)_{l+1} \leq (30)_{l+1}$  for  $l$  as in the above sum give the final estimate

$$|b_{i,j}^n| \leq C \frac{q^{j-i}}{\eta(i,j)} \left( 1 + \frac{12}{75}C \sum_{l=j}^{\infty} q^{2(l-j)} \right) = C \frac{q^{j-i}}{\eta(i,j)} \left( 1 + \frac{12}{75} \frac{C}{1-q^2} \right). \quad \square$$

Observe that the Main Theorem 3.1 is a corollary to Theorem 6.17. Thus the proof of Theorem 3.1 is completed.

## 7. APPLICATIONS

In the whole section, let  $\Delta = (s_i)_{i=1}^L$  be an arbitrary partition of the unit interval,  $k \geq 2$  an arbitrary integer and set  $\mu = (k, 1, 1, \dots, 1, 1, k)$ . Furthermore,  $\sigma = (k-1, \dots, k-1)$  and we denote by  $P_\Delta$  the orthogonal projection operator from  $L^2[0, 1]$  onto  $\mathbf{S}_k(\Delta; \sigma)$ . Additionally we assume that the B-splines  $N_i \equiv N_{i,k}$  are constructed as in Section 2.2 and their Gram matrix and its inverse are denoted by  $A = (a_{i,j})_{i,j=1}^M$  and  $B = (b_{i,j})_{i,j=1}^M$ , respectively. Moreover, we define the *Dirichlet kernel*  $K$  corresponding to the space  $\mathbf{S}_k(\Delta, \sigma)$  as

$$(7.1) \quad K(t, s) = \sum_{i,j=1}^M b_{i,j} N_i(t) N_j(s).$$

For an arbitrary subset  $U$  of the unit interval  $[0, 1]$ , we set  $\Delta(U) := \Delta \cap U$  understanding the partition  $\Delta$  as a subset of the unit interval  $[0, 1]$ . Using  $\Delta = (s_i)_{i=1}^L$ , we define  $|\Delta(U)| := \max \{s_{i+1} - s_i : 1 \leq i \leq L-1, \{s_i, s_{i+1}\} \subset U\}$ .

Let  $f$  be an integrable function on the unit interval  $[0, 1]$ . We define the *maximal function*  $M(f)$  of  $f$  as

$$(7.2) \quad M(f)(x) = \sup_{I \ni x} |I|^{-1} \int_I |f(t)| dt \quad \text{for all } x \in [0, 1],$$

and the *maximal average function*  $f^*$  of  $f$  as

$$(7.3) \quad f^*(x) = \sup_{I \ni x} \left| |I|^{-1} \int_I f(t) dt \right| \quad \text{for all } x \in [0, 1],$$

where in both cases, the supremum is taken over all subintervals  $I$  of  $[0, 1]$  containing  $x$ . Clearly,  $f^*(x) \leq Mf(x)$  for all  $x \in [0, 1]$ .

A point  $a \in [0, 1]$  is called a *weak Lebesgue point* of a locally integrable function  $f$  if

$$(7.4) \quad f(a) = \lim_{h \rightarrow 0} \int_a^{a+h} f(x) dx.$$



The Lebesgue differentiation theorem states in particular that a.e. point—with respect to Lebesgue measure—is a weak Lebesgue point.

We remember the notation from Example 2.6, where the function  $\eta_1 : \{1, \dots, M\} \times \{1, \dots, M\} \rightarrow \mathbb{R}^+$  was defined by

$$\eta_1(i, j) = \max_{i \wedge j \leq l \leq i \vee j} (k, 0)_l.$$

We shall need the following partial integration lemma due to Taberski [Tab62].

**Lemma 7.1** ([Tab62]). *Let  $f \in L^1[a, b]$  and  $g$  be a function of finite total variation on the interval  $[a + \varepsilon, b]$  for each  $\varepsilon > 0$  with*

$$(7.5) \quad \int_a^b \text{var}(g; [s, b]) \, ds < \infty.$$

*Here and in the following,  $\text{var}(f; [a, b])$  denotes the total variation of  $f$  on the interval  $[a, b]$ .*

*Under these assumptions, the following limit exists and we have*

$$(7.6) \quad \left| \lim_{\varepsilon \rightarrow 0} \int_{a+\varepsilon}^b f(t)g(t) \, dt \right| \leq \sup_{0 < h \leq b-a} \left| \frac{1}{h} \int_a^{a+h} f(t) \, dt \right| \int_a^b (|g(b)| + \text{var}(g; [s, b])) \, ds.$$

The theorems in this section are stated for general  $k$  under the hypothesis that the inverse Gramian  $B = (b_{i,j})_{i,j=1}^M$  satisfies a geometric decay inequality of the form

$$|b_{i,j}| \leq C_1 \frac{q^{|i-j|}}{\eta_1(i, j)}.$$

for some constants  $0 < q < 1, C_1 > 0$ . In Theorem 3.1 we proved such a bound with constants  $q, C_1$  independent of the underlying partition  $\Delta$  for  $k = 3$ . This allows us to apply each result below for this choice of  $k$ , that is for piecewise quadratic splines.

First, we present an ad hoc argument in the proof of Theorem 7.2 that majorizes  $P_\Delta f$  pointwise by the maximal function  $M(f)$ . This can be used to show that  $P_\Delta f$  converges back to  $f$  a.e. for every integrable function  $f$ , provided  $|\Delta| \rightarrow 0$ .

**Theorem 7.2.** *Suppose that*

$$(7.7) \quad |b_{i,j}| \leq C_1 \frac{q^{|i-j|}}{\eta_1(i, j)} \quad \text{for every } 1 \leq i, j \leq M,$$

*where  $0 < q < 1$  and  $C_1 > 0$  are some constants independent of  $i, j$ . Then we have that*

$$(7.8) \quad |P_\Delta f(t)| \leq C_2 M(f)(t)$$

*for all  $f \in L^1[0, 1]$  and  $t \in [0, 1]$ , where  $C_2 = C_1 \frac{1+2q-q^2}{(1-q)^2}$  is independent of  $f$  and  $t$ .*

*Proof.* Let  $t \in [0, 1]$ . We first note that  $P_\Delta$  is given by

$$(7.9) \quad P_\Delta f(t) = \sum_{i,j=1}^M b_{i,j} \langle f, N_j \rangle N_i(t).$$

Observe that

$$(7.10) \quad \begin{aligned} (1 + |j - i|)\eta_1(i, j) &= (1 + |j - i|) \max_{i \wedge j \leq l \leq i \vee j} (k, 0)_l \\ &\geq \sum_{l=i \wedge j}^{i \vee j} (k, 0)_l \geq t_{(i \vee j)+k} - t_{i \wedge j}. \end{aligned}$$

Now we choose the integer  $i_0$  such that  $t \in [t_{i_0}, t_{i_0+1})$  if  $t \in [0, 1)$  and we set  $i_0 = M$  if  $t = 1$ . Then we have by (7.9) and inequality (7.7)

$$\begin{aligned} |P_\Delta f(t)| &= \left| \sum_{i=i_0-k+1}^{i_0} N_i(t) \sum_{j=1}^M b_{i,j} \langle f, N_j \rangle \right| \\ &\leq \sum_{i=i_0-k+1}^{i_0} N_i(t) \sum_{j=1}^M C_1 \frac{q^{|i-j|}}{\eta_1(i, j)} |\langle f, N_j \rangle|. \end{aligned}$$

Using (7.10) we see that

$$(7.11) \quad |P_\Delta f(t)| \leq \sum_{i=i_0-k+1}^{i_0} N_i(t) \sum_{j=1}^M C_1 \frac{(|j - i| + 1)q^{|i-j|}}{t_{(i \vee j)+k} - t_{i \wedge j}} \int_{t_j}^{t_{j+k}} N_j(s) |f(s)| ds.$$

Since  $N_j(s) \leq 1$  pointwise and  $t \in [t_{i \wedge j}, t_{(i \vee j)+k}]$  if  $i$  and  $j$  are in the range of the sums in (7.11), this is less or equal

$$(7.12) \quad \sum_{i=i_0-k+1}^{i_0} N_i(t) \sum_{j=1}^M C_1 (|j - i| + 1) q^{|i-j|} M f(t).$$

By Proposition 2.4, the B-splines  $N_i$  are a partition of unity, so it follows that

$$(7.13) \quad |P_\Delta f(t)| \leq \max_{1 \leq i \leq M} \sum_{j=1}^M C_1 (|j - i| + 1) q^{|i-j|} M f(t).$$

An elementary calculation gives  $\sum_{j=1}^\infty (1 + |i - j|) q^{|i-j|} \leq \frac{1+2q-q^2}{(1-q)^2}$  for all  $1 \leq i < \infty$ . Consequently, we obtain the assertion of the theorem.  $\square$

**Remark 7.3.** If  $f \in C[0, 1]$ ,  $P_\Delta f \rightarrow f$  uniformly as  $|\Delta| \rightarrow 0$ . This is a consequence of the uniform boundedness of  $P_\Delta$  as an operator from  $C[0, 1]$  to  $C[0, 1]$  for fixed spline order  $k$  (proved in [Sha01]). Moreover,  $C[0, 1]$  is dense in  $L^1[0, 1]$  and  $M$  is of weak type  $(1, 1)$ . By Theorem 3.1, we may apply Theorem 7.2 to deduce  $|P_\Delta f(t)| \leq C_2 M(f)(t)$  for  $k = 3$  and all  $f \in L^1[0, 1]$ , where  $C_2$  is independent of  $\Delta$  and  $f$ . From these facts, we obtain by a standard argument

$$\lim_{|\Delta| \rightarrow 0} P_\Delta f(x) = f(x), \quad \text{for a. e. } x \in [0, 1], f \in L^1[0, 1] \text{ and } k = 3.$$

In fact, this property is local in the sense that for every open subset  $U \subset [0, 1]$  we have

$$\lim_{|\Delta(U)| \rightarrow 0} P_\Delta f(x) = f(x), \quad \text{for a. e. } x \in U, f \in L^1[0, 1] \text{ and } k = 3.$$

We will refine these results in the following in the sense that we bound  $P_\Delta f$  pointwise by the maximal average function  $f^*$  and show convergence of  $P_\Delta f(a)$  to  $f(a)$  as  $|\Delta| \rightarrow 0$  for all weak Lebesgue points  $a$  of  $f$ .

The proofs of the theorems below are taken basically from [CK97], where these results were shown for piecewise linear splines.

The following theorem, concerning the total variation of the Dirichlet kernel, is essential.

**Theorem 7.4.** *Suppose that*

$$(7.14) \quad |b_{i,j}| \leq C_1 \frac{q^{|i-j|}}{\eta_1(i,j)} \quad \text{for every } 1 \leq i, j \leq M,$$

where  $0 < q < 1$  and  $C_1 > 0$  are some constants independent of  $i, j$ . Then, for all  $t \in (0, 1)$ , we have

$$(7.15) \quad \max \left\{ \int_t^1 \text{var}(K(t, \cdot); [s, 1]) \, ds, \int_0^t \text{var}(K(t, \cdot); [0, s]) \, ds \right\} \leq C_3,$$

where  $C_3 = \frac{2kq^{-k+1}C_1}{(1-q)^2}$  and  $\text{var}(f; [a, b])$  denotes the total variation of  $f$  on the interval  $[a, b]$ .

*Proof.* We prove the inequality for the first expression in the maximum. The proof for the second expression is similar. We first fix  $t \in [0, 1)$  and choose the integer  $i_0$  such that  $t \in [t_{i_0}, t_{i_0+1})$ . Then we have by elementary properties of the total variation

$$(7.16) \quad \begin{aligned} \int_t^1 \text{var}(K(t, \cdot); [s, 1]) \, ds &\leq \int_{t_{i_0}}^1 \text{var}(K(t, \cdot); [s, 1]) \, ds \\ &= \sum_{i=i_0}^M \int_{t_i}^{t_{i+1}} \text{var}(K(t, \cdot); [s, 1]) \, ds \\ &\leq \sum_{i=i_0}^M \int_{t_i}^{t_{i+1}} \text{var}(K(t, \cdot); [t_i, 1]) \, ds \\ &\leq \sum_{i=i_0}^M (10)_i \sum_{j=i}^M \text{var}(K(t, \cdot); [t_j, t_{j+1}]). \end{aligned}$$

We now observe that the total variation of B-spline functions  $N_j$  is  $\leq 2$ . This is a consequence of the following identity for derivatives of B-splines

$$N'_{i,k}(x) = (k-1) \left( \frac{N_{i+1,k-1}(x)}{(k, 1)_i} - \frac{N_{i,k-1}(x)}{(k-1, 0)_i} \right),$$

since we have by this formula and Proposition 2.4

$$\begin{aligned} \text{var}(N_{i,k}; \text{supp } N_{i,k}) &= \int_{t_i}^{t_{i+k}} |N'_{i,k}(t)| \, dt \\ &= (k-1) \int_{t_i}^{t_{i+k}} \left| \frac{N_{i+1,k-1}(t)}{(k, 1)_i} - \frac{N_{i,k-1}(t)}{(k-1, 0)_i} \right| \, dt \end{aligned}$$

$$\begin{aligned}
&\leq \frac{k-1}{(k,1)_i} \|N_{i+1,k-1}\|_{L^1(0,1)} + \frac{k-1}{(k-1,0)_i} \|N_{i,k-1}\|_{L^1(0,1)} \\
&= 2.
\end{aligned}$$

We use this bound on the total variation of  $N_{i,k}$  to estimate the total variation of the Dirichlet kernel appearing in (7.16) as

$$\text{var}(K(t, \cdot); [t_j, t_{j+1}]) \leq 2 \sum_{l=i_0-k+1}^{i_0} \sum_{m=j-k+1}^j |b_{l,m}| N_l(t)$$

The fact that the B-splines are a partition of unity and inequality (7.14) give us

$$\begin{aligned}
(7.17) \quad \text{var}(K(t, \cdot); [t_j, t_{j+1}]) &\leq 2 \sum_{l=i_0-k+1}^{i_0} \sum_{m=j-k+1}^j |b_{l,m}| N_l(t) \\
&\leq 2k \max_{\substack{i_0-k+1 \leq l \leq i_0 \\ j-k+1 \leq m \leq j}} |b_{l,m}| \\
&\leq 2kC_1 \max_{\substack{i_0-k+1 \leq l \leq i_0 \\ j-k+1 \leq m \leq j}} \frac{q^{|m-l|}}{\eta_1(l, m)} \\
&\leq 2kC_1 q^{-k+1} q^{j-i_0} \max_{\substack{i_0-k+1 \leq l \leq i_0 \\ j-k+1 \leq m \leq j}} \eta_1(l, m)^{-1}.
\end{aligned}$$

Observe that  $(10)_i \leq \eta_1(l, m)$  for all  $l, m$  in the range of the above maximum and  $j \geq i \geq i_0$ . Thus we estimate the last line of (7.16) using (7.17) by

$$(7.18) \quad 2kq^{-k+1}C_1 \sum_{i=i_0}^M \sum_{j=i}^M q^{j-i_0} \leq \frac{2kq^{-k+1}C_1}{(1-q)^2}.$$

This concludes the proof of the theorem.  $\square$

**Lemma 7.5.** *Suppose that*

$$(7.19) \quad |b_{i,j}| \leq C_1 \frac{q^{|i-j|}}{\eta_1(i, j)} \quad \text{for every } 1 \leq i, j \leq M,$$

where  $0 < q < 1$  and  $C_1 > 0$  are some constants independent of  $i, j$ . Let  $a, s \in [0, 1)$  and define the integers  $i_0, j_0$  such that  $a \in [t_{i_0}, t_{i_0+1})$  and  $s \in [t_{j_0}, t_{j_0+1})$ . Then the Dirichlet kernel  $K$  satisfies the bound

$$(7.20) \quad |K(a, s)| \leq C_1 q^{-k+1} \frac{q^{|i_0-j_0|}(1+|i_0-j_0|)}{|s-a|}.$$

*Proof.* Since the B-splines are a partition of unity and due to estimate (7.19) we have

$$\begin{aligned}
(7.21) \quad |K(a, s)| &= \left| \sum_{l,m=1}^M b_{l,m} N_l(s) N_m(a) \right| \leq \max_{\substack{i_0-k+1 \leq l \leq i_0 \\ j_0-k+1 \leq m \leq j_0}} |b_{l,m}| \\
&\leq C_1 \max_{\substack{i_0-k+1 \leq l \leq i_0 \\ j_0-k+1 \leq m \leq j_0}} \frac{q^{|l-m|}}{\eta_1(l, m)}.
\end{aligned}$$

Next we observe for  $l, m$  in the range of the maximum in (7.21):

$$(7.22) \quad \eta_1(l, m) \geq \max_{i_0 \wedge j_0 \leq u \leq i_0 \vee j_0} (10)_u \geq \frac{|s - a|}{|i_0 - j_0| + 1} \quad \text{and} \\ |l - m| \geq |i_0 - j_0| - k + 1.$$

This and (7.21) allow us to deduce

$$(7.23) \quad |K(a, s)| \leq C_1 q^{-k+1} \frac{q^{|i_0 - j_0|} (1 + |i_0 - j_0|)}{|s - a|}.$$

This inequality proves the lemma.  $\square$

**Theorem 7.6.** *Suppose that*

$$(7.24) \quad |b_{i,j}| \leq C_1 \frac{q^{|i-j|}}{\eta_1(i, j)} \quad \text{for every } 1 \leq i, j \leq M,$$

where  $0 < q < 1$  and  $C_1 > 0$  are some constants independent of  $i, j$ . Then we have the inequality

$$|P_\Delta f(x)| \leq C_4 f^*(x)$$

for all  $f \in L^1(0, 1)$  and  $x \in (0, 1)$ , where  $C_4 = 2(C_3 + C_1 q^{-k+1} \max_{1 \leq i \leq M} q^i (1+i))$  is independent of  $f$  and  $x$ . Here, the constant  $C_3$  is the same as in Theorem 7.4

*Proof.* Let  $t \in (0, 1)$  and  $f \in L^1(0, 1)$ . We first note that

$$(7.25) \quad P_\Delta f(t) = \left( \int_0^t + \int_t^1 \right) K(t, s) f(s) \, ds.$$

Lemma 7.1 yields that

$$\left| \int_t^1 K(t, s) f(s) \, ds \right| \leq \sup_{0 < h \leq 1-t} \frac{1}{h} \left| \int_t^{t+h} f(s) \, ds \right| \\ \cdot \int_t^1 (|K(t, 1)| + \text{var}(K(t, \cdot); [s, 1])) \, ds.$$

We use now Theorem 7.4, Lemma 7.5 and the definition of the maximal average function  $f^*$  to deduce

$$\left| \int_t^1 K(t, s) f(s) \, ds \right| \leq (C_1 q^{-k+1} \max_{1 \leq i \leq M} q^i (1+i) + C_3) f^*(t).$$

By a symmetric argument, we get the same inequality for the second integral  $\int_0^t K(t, s) f(s) \, ds$  in (7.25), and thus the theorem is proved.  $\square$

**Theorem 7.7.** *Let  $f \in L^1(0, 1)$  and  $a$  be a weak Lebesgue point of  $f$ . Additionally, let  $U$  be an arbitrary neighbourhood of  $a$ . Suppose that*

$$(7.26) \quad |b_{i,j}| \leq C_1 \frac{q^{|i-j|}}{\eta_1(i, j)} \quad \text{for every } 1 \leq i, j \leq M,$$

where  $0 < q < 1$  and  $C_1 > 0$  are some constants independent of  $i, j$  and  $\Delta$ . Then we have

$$(7.27) \quad \lim_{|\Delta(U)| \rightarrow 0} P_\Delta f(a) = f(a).$$

*Proof.* We assume without loss of generality that  $U$  is an open interval containing  $a$ . We remark that  $\int_0^1 K(t, s) ds = 1$  for all  $t \in [0, 1]$ , thus we have to estimate

$$(7.28) \quad P_\Delta f(a) - f(a) = \int_0^1 K(a, s)(f(s) - f(a)) ds.$$

Let  $\varepsilon > 0$  be an arbitrary positive real number. Since  $a$  is a weak Lebesgue point of  $f$ , there exists  $\delta > 0$  such that

$$(7.29) \quad M = \sup_{0 < |h| \leq \delta} \left| \frac{1}{h} \int_a^{a+h} f(t) - f(a) dt \right| < \varepsilon \quad \text{and} \quad \{s : |s - a| < \delta\} \subset U.$$

We now split the integral in (7.28) into the three parts

$$(7.30) \quad \int_{|s-a|>\delta} + \int_a^{a+\delta} + \int_{a-\delta}^a.$$

We begin with the first integral  $\int_{|s-a|>\delta}$ :

$$(7.31) \quad \left| \int_{|s-a|>\delta} \right| \leq \|f\|_1 \sup_{|s-a|>\delta} |K(a, s)| + |f(a)| \int_{|s-a|>\delta} |K(a, s)| ds.$$

If  $|\Delta(U)| \rightarrow 0$ , the expression  $\sup_{|s-a|>\delta} |K(a, s)|$  converges to zero by Lemma 7.5. Now we estimate the integral  $\int_{|s-a|>\delta} |K(a, s)| ds$ . It is enough to consider the integral  $\int_{a+\delta}^1 |K(a, s)| ds$ , since the other part follows by symmetric arguments. First we let the indices  $i_0, j_0$  be such that  $a \in [t_{i_0}, t_{i_0+1})$  and  $a + \delta \in [t_{j_0}, t_{j_0+1})$ . Then we conclude by (7.21) and (7.22)

$$\begin{aligned} \int_{a+\delta}^1 |K(a, s)| ds &\leq \sum_{l=j_0}^M \int_{t_l}^{t_{l+1}} |K(a, s)| ds \\ &\leq C_1 \sum_{l=j_0}^M \frac{(10)_l}{\max_{i_0 \leq u \leq l} (10)_u} q^{-k+1+|l-i_0|} \\ &\leq C_1 q^{-k+1} \sum_{l=j_0}^{\infty} q^{|l-i_0|}. \end{aligned}$$

The right hand side of the above inequality converges to zero, provided  $|\Delta(U)| \rightarrow 0$ , so we have proved that

$$(7.32) \quad \int_{|s-a|>\delta} |K(a, s)| ds \rightarrow 0 \quad \text{if } |\Delta(U)| \rightarrow 0.$$

In the second step, we estimate the integral  $\int_a^{a+\delta} K(a, s)(f(s) - f(a)) ds$ . The third integral  $\int_{a-\delta}^a K(a, s)(f(s) - f(a)) ds$  is treated in the same way. We apply Lemma 7.1 to this integral with the choice  $a = a$ ,  $b = a + \delta$ ,  $f$  replaced by the

function  $f(\cdot) - f(a)$  and  $g(t) = K(a, t)$ . This yields

$$(7.33) \quad \left| \int_a^{a+\delta} K(a, s)(f(s) - f(a)) \, ds \right| \leq \sup_{0 < h \leq \delta} \frac{1}{h} \left| \int_a^{a+h} (f(t) - f(a)) \, dt \right| \cdot \int_a^{a+\delta} |K(a, a + \delta)| + \text{var}(K(a, \cdot); [t, a + \delta]) \, dt.$$

By (7.29),  $\sup_{0 < h \leq \delta} \frac{1}{h} \left| \int_a^{a+h} f(t) - f(a) \, dt \right| \leq \varepsilon$ . Theorem 7.4 yields

$$\int_a^{a+\delta} \text{var}(K(a, \cdot); [t, a + \delta]) \, dt \leq C_3,$$

and Lemma 7.5 gives

$$\delta |K(a, a + \delta)| \leq C_1 q^{-k+1} q^{|i_0 - j_0|} (1 + |i_0 - j_0|).$$

Inserting these facts in (7.33), we get

$$(7.34) \quad \left| \int_a^{a+\delta} K(a, s)(f(s) - f(a)) \, ds \right| \leq C\varepsilon$$

for some constant  $C$  independent of  $\Delta$  and  $f$ .

Combining (7.32) with (7.34) in the splitting (7.30), we see that  $|P_\Delta f(a) - f(a)|$  is arbitrarily small, provided  $|\Delta(U)|$  is chosen sufficiently small. This proves the theorem.  $\square$

**Remark 7.8.** As a last remark we note that Theorem 3.1 can also be used to prove unconditionality for orthogonal spline systems of order 3 with arbitrary knots in  $L^p$ ,  $1 < p < \infty$ . This will be investigated in a future paper.

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INSTITUTE OF ANALYSIS, JOHANNES KEPLER UNIVERSITY LINZ, AUSTRIA, 4040 LINZ,  
ALTENBERGER STRASSE 69

*E-mail address:* markus.passenbrunner@jku.at